POLYTECH
PARIS-SUD

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Cycle Ingénieur – 2ème année
Département Informatique

Verification and Validation

Part IV :

Deductive Verification (I)

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How to do Verification ?

In the sequel, we concentrate on

Deductive Verification

(Proof Techniques)

Motivation: Hoare - Logic

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A Logic for Programs ???

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- We consider the Hoare-Logic, technically an inference system PL + E + A + Hoare

Recall: Proof Systems

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- Assumptions and conclusions are terms in a logic containing variables

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$$\frac{1+2 = a \quad a = 3}{1+2 = 3}$$

{x↦τ*5, y↦5*τ}

$$\frac{\tau * 5 = 5 * \tau \quad 5 * \tau = z}{\tau * 5 = z}$$

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$$
\dfrac{\dfrac{\dfrac{f(a,b)=a}{a=f(a,b)} \qquad \dfrac{f(a,b)=a \quad f(f(a,b),b)=c}{f(a,b)=c}}{a=c} \qquad \dfrac{}{g(a)=g(a)}}{g(a)=g(c)}
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$$\dfrac{a=c \qquad g(a)=g(a)}{g(a)=g(c)}$$

- The non-elementary facts at the leaves are the *global assumptions* (here $f(a,b) = a$ and $f(f(a,b),b) = c$).

Hoare - Logic: A Proof System for Programs

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  - the assignment $x := = E \quad (x \in V)$
  - the sequential compos. $c_1 \; ; \; c_2$
  - the conditional IF cond THEN c_1 ELSE c_2
  - the loop WHILE cond DO c

  where c, c_1, c_2, are cmd's, V variables,

  E an arithmetic expression, and cond a boolean expression.

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  - *P* and *Q* are formulas over the variables *V*, so they can be seen as set of possible states.

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If a program *cmd* starts in a state admitted by *P* if it terminates, that the program must reach a state that satisfies *P*.

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$$\frac{P \rightarrow P' \quad \vdash \{P'\}\ cmd\ \{Q'\} \quad Q' \rightarrow Q}{\vdash \{P\}\ cmd\ \{Q\}}$$

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Let's consider it one by one …

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well, states do not change ...

Therefore, valid states remain valid.

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- The assignment rule:

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$$\frac{}{\vdash \{P[x \mapsto E]\}\ \mathrm{x} := \mathrm{E} \{P\}}$$

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- Example (1):

$\vdash \{1 \leq x \wedge x \leq 10\}$ x:== x+2 $\{3 \leq x \wedge x \leq 12\}$

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$\equiv 3 \leq (x+2) \wedge (x+2) \leq 12$

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$\equiv 1 \leq x \wedge x \leq 10$

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≡ 2=2 ≡ true (reflexivity…)

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The conditional-rule:

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Example (3):

$$\vdash \{true\}\ \text{IF}\ 0 \leq x\ \text{THEN SKIP ELSE}\ x :== -x\ \{0 \leq x\}$$

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This can be extended to the formal proof:

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- The *sequence* rule:

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- essentially a relational composition on state sets.

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0

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The rule for the sequence.

Example (4):

0

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Example (4):

$$\frac{}{\vdash \{true\}\ tm :== 1; (sum :== 1; i :== 0)\ \{tm = 1 \land sum = 1 \land i = 0\}}$$

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where $A = tm = 1 \land sum = 1 \land i = 0$ and where $B = tm = 1 \land sum = 1$.

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It is often practical to introduce abbreviations.

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- The while-rule.

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- This works like an induction: if some P is true after n traversals of the loop and remain true for the n+1 traversal, it must be always true.

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- The predicate P is called an invariant. Note that an invariant can be maintained even if the concrete state changes ! See:

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$\vdash$ {1≤x ∧ x≤10} WHILE x < 10 DO x:== x+1 {¬ (x < 10) ∧ 1≤x ∧ x≤10}

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$$\frac{P \to P' \quad \vdash \{P'\}\ cmd\ \{Q'\} \quad Q' \to Q}{\vdash \{P\}\ cmd\ \{Q\}}$$

Reflects the intuition that P' is a subset of legal states P and Q is a subset of legal states Q'.

Hoare - Logic: A Proof System for Programs

- The consequence-rule:

$$\frac{P \rightarrow P' \quad \vdash \{P'\}\ cmd\ \{Q'\} \quad Q' \rightarrow Q}{\vdash \{P\}\ cmd\ \{Q\}}$$

Reflects the intuition that P' is a subset of legal states P and Q is a subset of legal states Q'.

This is the only rule that is not determined by the syntax of the program; it can be applied anywhere in the (Hoare-) proof.

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Example (5) (the continuation of Example (3)):

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Example (5) (the continuation of Example (3)):

$$\frac{true \land \neg(0 \leq x) \to (0 \leq -x) \quad \dfrac{}{\vdash \{(0 \leq x)[x \mapsto -x]\}\ x :== -x\ \{0 \leq x\}} \quad 0 \leq x \to 0 \leq x}{\vdash \{true \land \neg(0 \leq x)\}\ x :== -x\ \{0 \leq x\}}$$

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- The Hoare calculus has a number of implicit consequences, i.e. rules that can be derived from the other ones.

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- A handy derived rule, the False-rule:

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- Another handy corollary of the False-rule:

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$$\frac{}{\vdash \{P \land \neg cond\}\ \mathrm{WHILE}\ cond\ \mathrm{DO}\ c\ \{P \land \neg cond\}}$$

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- Another handy corollary of the False-rule:

$$\frac{}{\vdash \{P \wedge \neg cond\}\ \text{WHILE}\ cond\ \text{DO}\ c\ \{P \wedge \neg cond\}}$$

Proof:

by consequence-rule, while-rule,

P and cond-negation,

False-rule.

This means: If we can not enter into the WHILE-loop, it behaves like a SKIP.

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- Yet another handy corollary of the consequence rule:

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- Yet another handy corollary of the consequence rule:

$$\frac{P = P' \quad \vdash \{P'\}\ cmd\ \{Q'\} \quad Q' = Q}{\vdash \{P\}\ cmd\ \{Q\}}$$

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- *Note: We will allow to apply this rule implicitly, thus leveraging local "logical massage" of pre- and post-conditions.*

Hoare - Logic: A Proof System for Programs

- Example (6):

$$\frac{}{\vdash \{true\}\ \text{WHILE}\ true\ \text{DO}\ SKIP\ \{x = 42\}}$$

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true ∧ ¬*true* ≡ *false*

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Hoare - Logic: A Proof System for Programs

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<u>Note</u>:

Hoare-Logic is a calculus for

partial correctness; for non-terminating

programs, it is possible to prove *anything*!

Hoare - Logic: A Proof System for Programs

- Example (7):

$$\frac{}{\vdash \{true\}\ \text{WHILE}\ x < 2\ \text{DO}\ x :== x + 1\ \{2 \leq x\}}$$

Hoare – Logic: A Proof System for Programs

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We can't apply the WHILE-rule directly — the only other choice is the consequence rule. Instantiating the invariant variable P by a fresh variable I allows us to bring the triple into a shape that we can apply the WHILE rule later

Hoare - Logic: A Proof System for Programs

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Now we can apply the while rule.

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To be sure (entering the while loop) we apply again the consequence rule. For the missing bit, we instantiate I".

Hoare - Logic: A Proof System for Programs

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Now, in order to make the assignment rule "fit", we must have $I'' \equiv I'[x \mapsto x+1]$.

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Additionally, in order that this constitutes a Hoare-Proof, we must have all the implications.

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Hoare - Logic: A Proof System for Programs

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So, we have a Hoare Proof iff we have a solution to the following list of constraints:

$I'' \equiv I'[x \mapsto x+1]$

$A \equiv true \rightarrow I$

$B \equiv I \wedge \neg(x < 2) \rightarrow 2 \leq x$

$C \equiv I \wedge x < 2 \rightarrow I'[x \mapsto x+1]$

Hoare - Logic: A Proof System for Programs

- Example (7):

  Proof:

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$$B \equiv I \wedge \neg(x < 2) \rightarrow 2 \leq x$$

$$C \equiv I \wedge x < 2 \rightarrow I'[x \mapsto x+1]$$

$$D = I' \rightarrow I$$

  - I must be *true*, this solves A, B, D
  - we are fairly free for a solution for I'; e.g. $x \leq 2$ or $x \leq 5$ would do the trick !

Hoare - Logic: Some facts.

Assume that we have a reasonably well-defined "compiler function" that maps a program to a relation from input to output states:

$$C : \text{cmd} \rightarrow (\sigma\times\sigma)\text{Set}$$

(See Winskell's Book)

Then we can define the "validity" of a specification:

$$\models \{P\}\ cmd\ \{Q\} \equiv \forall \sigma, \sigma'.(\sigma, \sigma') \in C(cmd) \rightarrow P(\sigma) \rightarrow Q(\sigma')$$

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  - … provided we have solutions for the invariants.

Hoare - Logic: Some facts.

Theorem: Correctness of the Hoare-Calculus:

$$\vdash \{P\}\ cmd\ \{Q\} \rightarrow\ \models \{P\}\ cmd\ \{Q\}$$

... so, whenever there is a proof, it is also valid wrt. C.

Hoare - Logic: Some facts.

Theorem: Relative Completeness of the Hoare-Calculus

$$\models \{P\}\ cmd\ \{Q\} \ \rightarrow\ \vdash \{P\}\ cmd\ \{Q\}$$

Amazingly, this holds also the other way round:
whenever a specification is valid, (and we can solve all the implications on arithmetics), there is a Hoare-Proof.

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- Another Example (8) : The integer square-root

Hoare - Logic: A Proof System for Programs

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```
int i = 0;
int tm = 1;
int sum = 1;
WHILE sum ≤ a DO
  i  := i+1;
  tm := tm + 2;
  sum:= sum + tm;
```

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```
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- Program and Specification in a Hoare Triple

$$\vdash \{a \geq 0\}\ \text{prelude; WHILE sum} \leq a\ \text{DO body}\ \{\text{post}\}$$

where $\text{post} \equiv i^2 \leq a \wedge a < (i+1)^2$

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$$\frac{\dfrac{}{\vdash \{true\} tm :== 1 \{tm = 1\}} \qquad \dfrac{\dfrac{}{\vdash \{tm = 1\} sum :== 1 \{B\}} \quad \dfrac{}{\vdash \{B\}\ i :== 0\ \{A\}}}{\vdash \{tm = 1\}\ sum :== 1; i :== 0\ \{A\}}}{\vdash \{true\}\ tm :== 1; (sum :== 1; i :== 0)\ \{tm = 1 \wedge sum = 1 \wedge i = 0\}}$$

We know that already ....

where $A = tm = 1 \wedge sum = 1 \wedge i = 0$ and where $B = tm = 1 \wedge sum = 1$.

  - and:

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$I'' \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2]$

$I \wedge sum \leq a \longrightarrow I'$

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$a < sum \wedge I \longrightarrow post$

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$I \wedge sum \leq a \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2][i \mapsto i+1]$ "Invariant is preserved in body"

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$a \geq 0 \wedge A \longrightarrow I$ "Invariant initially holds at loop entry"

Recall: $\ldots \equiv a \geq 0 \wedge i=0 \wedge tm=1 \wedge sum=1$

$a < sum \wedge I \longrightarrow post$ "Invariant at loop exit implies post"

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- Our proof boils further down to finding the invariant I

$I \wedge sum \leq a \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2][i \mapsto i+1]$

$a \geq 0 \wedge i=0 \wedge tm=1 \wedge sum=1 \longrightarrow I$

$a < sum \wedge I \longrightarrow i^2 \leq a \wedge a < (i+1)^2$

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- Our proof boils further down to finding the invariant I

$i \geq 0$

$I \wedge sum \leq a \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2][i \mapsto i+1]$

$a \geq 0 \wedge i=0 \wedge tm=1 \wedge sum=1 \longrightarrow I$

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$i \geq 0$

$tm \geq 1$

$a \geq 0 \wedge i=0 \wedge tm=1 \wedge sum=1 \longrightarrow I$

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$i \geq 0$

$tm \geq 1$

$sum \geq 1$

$tm = 2*i + 1$

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$sum = \sum_{k=0}^{i} (2*k + 1)$

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$a \geq i^2$

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- Our proof boils further down to finding the invariant I

$$I \wedge sum \leq a \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2][i \mapsto i+1]$$

$$a \geq 0 \wedge i=0 \wedge tm=1 \wedge sum=1 \longrightarrow I$$

$$a < sum \wedge I \longrightarrow i^2 \leq a \wedge a < (i+1)^2$$

$$I \equiv sum = (i+1)^2 \wedge a \geq i^2 \wedge tm = 2*i + 1 \wedge tm \geq 1$$

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- We check our invariant (1)

$I \wedge sum \leq a \longrightarrow I[sum \mapsto sum+tm][tm \mapsto tm+2][i \mapsto i+1]$

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Hoare - Logic: A Proof System for Programs

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Invariant preserved

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Invariant initially holds

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Invariant implies post-condition

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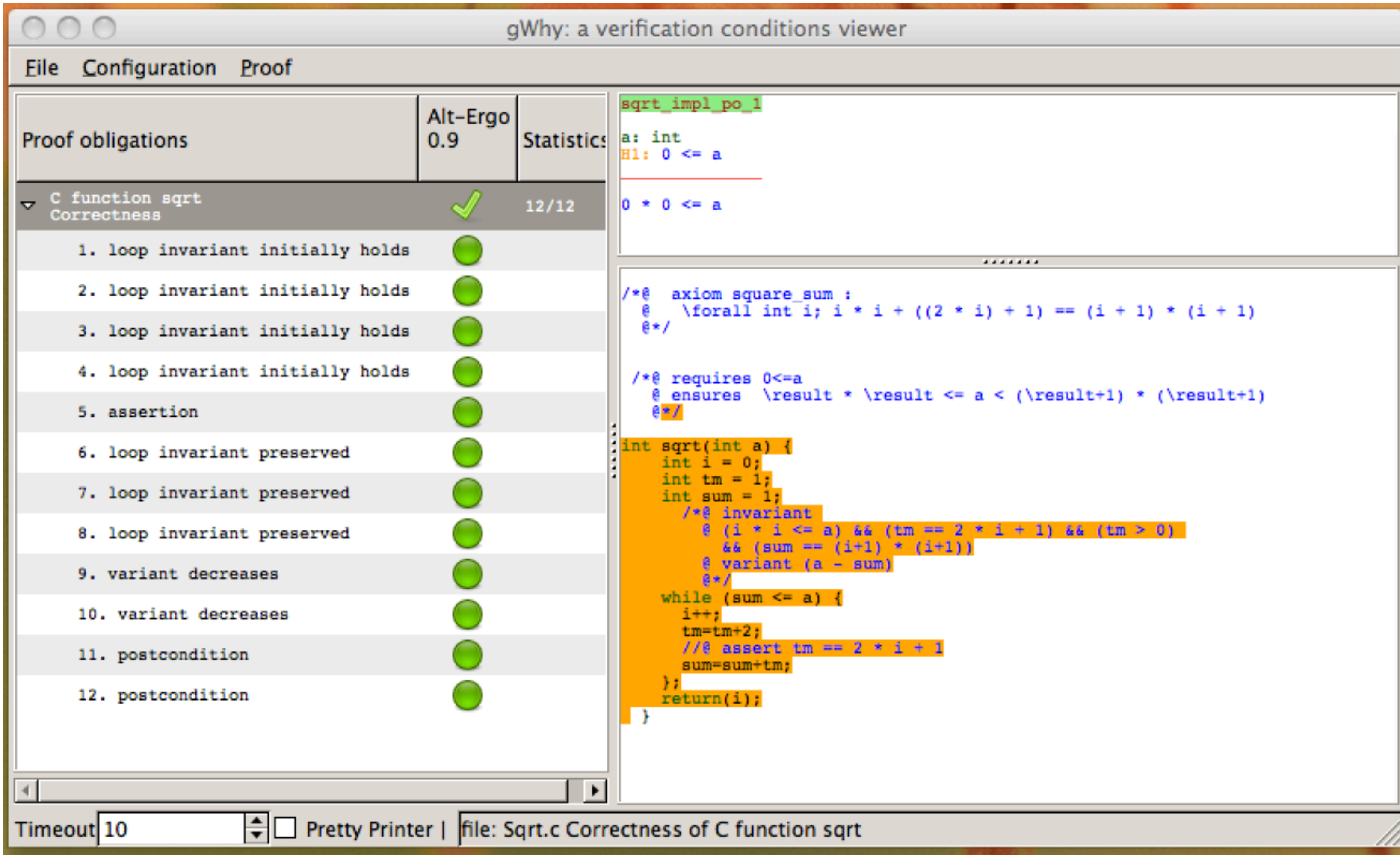
  - Iff such a function m (a measure) exists, the loop will terminate.
  - A candidate for m: $m(a, i, tm, sum) \equiv a - i$ which obviously decreases.

Tools: gwhy and Squareroot



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- … in the essence, the Hoare Calculus is an entirely syntactic game that constructs a labelling of the program with assertions …

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Formal Proof

- Can be very hard - up to infeasible (nobody will probably ever prove the correctness of MS Word!)

- But still, the proof-task can be automated to a large extent.

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- Limits in perfection:

  We know only up to a given "certainty" that the
  program meets the specification ...