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EXISTENCE OF NASH EQUILIBRIUM FOR CHANCE-CONSTRAINED GAMES

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Existence of Nash equilibrium for Chance-Constrained Games

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Abstract

We consider an n-player strategic game with finite action sets. The payoffs of each player are random variables. We assume that each player uses a satisficing payoff criterion defined by a chance-constraint, i.e., players face a chance-constrained game. We consider the cases where payoffs follow normal and elliptically symmetric distributions. For both cases we show that there always exists a mixed strategy Nash equilibrium of corresponding chance-constrained game.

Keywords: Chance-constrained game, Elliptically symmetric distribution, Normal distribution, Nash equilibrium.

1. Introduction

In 1928, John von Neumann [1] showed that there exists a mixed strategy saddle point equilibrium for a two player zero sum game with finite number of actions for each player. In 1950, John Nash [2] showed that there always exists a mixed strategy Nash equilibrium for an n-player general sum game with finite number of actions for each player. In both [1, 2], it is considered that the players' payoffs are deterministic. But, there can be the cases where the players' payoffs are random variables following certain distributions. The wholesale electricity markets are the good examples that capture this situation.

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A recent paper by Mazadi et al. [3] introduced wind integration on electricity markets due to which the payoffs become random variables. In some cases the consumers' demand are random that introduce randomness in the firms' payoffs [4, 5, 6]. One obvious way to handle such type of games is to replace random variables by their expected value and solve the corresponding deterministic game [5, 6]. But, this criterion fails to take proper account of the stochasticity in the payoffs. For example, the observed sample payoffs can be large amounts with very small probabilities and players may be satisfied to get payoffs of highest level with certain probability. To capture such a situation the concept of satisficing has been considered in the literature where players are interested in a strategy which maximizes their total payoff that can be obtained with at least a given probability. Such payoff criterion is defined using chance-constrained programming [7, 8, 9] and due to which we call such games as chance-constrained games. The papers [3, 4] mentioned above on electricity markets use chance-constrained game formulation to study the situation. The action sets of the players are not finite in both cases. In [3] the game problem is formulated as an equivalent linear complementarity problem (LCP). Hence, the existence of Nash equilibrium depends on whether the corresponding LCP has a solution. The existence of Nash equilibrium for the games where the action sets are not finite is not easy to show even when the payoffs are deterministic. It depends on the nature of action sets and payoff functions [10, 11]. There is also a game theoretic situation in electricity market where the action sets are finite [12]. Although the players' payoffs are deterministic in [12], the counterpart of the model where the payoffs are random variables using chance-constrained game formulation can be considered. Only few theoretical results on zero sum chance-constrained games with finite action sets of all players are available in the literature so far [13, 14, 15, 16].

In this paper we focus on the games where the payoffs of the players are random variables and each player considers a satisficing payoff criterion defined using a chance-constraint. The players face a chance-constrained game. To the best of our knowledge there is no result on the existence of Nash equilibrium of chance-constrained games even when the action sets of all players are finite. We consider an n-player game where payoffs of each player are random variables. We consider the cases where the payoffs are normal and elliptically symmetric random variables. For each case we show that there always exists a mixed strategy Nash equilibrium for the underlying chance-constrained game.

The structure of the rest of the paper is as follows: in Section 2 we give the definition of chance-constrained games. Existence of mixed strategy Nash equilibrium is then given in Section 3.

2. The Model

We consider an n-player strategic game. Let $I = \{1, 2, \cdots, n\}$ be a set of all players. For each $i \in I$, let A_i be a finite action set of player i and its generic element is denoted by a_i. A vector $a = (a_1, a_2, \cdots, a_n)$ denotes an action profile of the game. Let $A = \times_{i=1}^{n} A_i$ be a set of all action profiles of the game. Denote, $A_{-i} = \times_{j=1;j\neq i}^{n} A_i$ and $a_{-i} \in A_{-i}$ is a vector of actions a_j, $j \neq i$. The action set A_i of player i is also called as a set of pure strategies of player i. A mixed strategy of a player is represented by a probability distribution over its action set. For each $i \in I$, let X_i be a set of mixed strategies of player i, i.e., it is a set of all probability distributions over an action set A_i. A mixed strategy $\tau_i \in X_i$ is represented by $\tau_i = (\tau_i(a_i))_{a_i \in A_i}$, where $\tau_i(a_i) \geq 0$ is a probability with which player i chooses an action a_i and $\sum_{a_i \in A_i} \tau_i(a_i) = 1$. Let $X = \times_{i=1}^{n} X_i$ be a set of all mixed strategy profiles of the game and its element is denoted by $\tau = (\tau_i)_{i \in I}$. Denote, $X_{-i} = \times_{j=1;j\neq i}^{n} X_i$ and $\tau_{-i} \in X_{-i}$ is a vector of mixed strategies of all players excluding player i. We define (ν_i, τ_{-i}) to be a strategy profile where player i uses strategy ν_i and each player j, $j \neq i$, uses strategy τ_j. Let $r_i : A \to \mathbb{R}$ be a payoff function of player i. Specifically, player i gets payoff $r_i(a)$ when player i, $i \in I$, chooses an action a_i. For a given strategy profile $\tau \in X$ the payoff of player i, $i \in I$, is defined as

$$r_i(\tau) = \sum_{a \in A} \prod_{j=1}^{n} \tau_j(a_j) r_i(a). \tag{2.1}$$

For such games, Nash [2] showed that there always exists a Nash equilibrium in mixed strategies.

We consider the case where payoffs of each player are random variables and follow a certain distribution. One way to handle these games is by taking the expected value of random variables and solve the corresponding deterministic game. Another way to deal with this situation is by using satisficing payoff criterion defined by a chance constraint [13, 14, 15, 16]. We assume that each player uses satisficing payoff criterion, i.e., at strategy profile $\tau \in X$ the payoff of each player is the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori and it is not known to the other players. Let $\alpha_i \in [0,1]$ be the confidence level of player i and $\alpha = (\alpha_i)_{i \in I}$. For a given strategy profile $\tau \in X$ and a confidence level vector α the payoff of player i, $i \in I$, is given by

$$u_i^{\alpha_i}(\tau) = \sup\{u | P(r_i(\tau) \geq u) \geq \alpha_i\}. \tag{2.2}$$

These games are known as chance-constrained games because the payoff of each player is defined using a chance-constraint. For a given $\alpha \in [0,1]^n$, the above chance constrained game is a non-cooperative game where the payoffs of a player defined by (2.2) is known to all other players. The set of best response strategies of player i, $i \in I$, against a given strategy profile τ_{-i} of other players is given by

$$BR_i^{\alpha_i}(\tau_{-i}) = \{\bar{\tau}_i \in X_i | u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}),\ \forall\ \tau_i \in X_i\}. \tag{2.3}$$

Next, we give the definition of Nash equilibrium.

Definition 2.1 (Nash equilibrium). *A strategy profile $\tau^* \in X$ is said to be a Nash equilibrium for a given $\alpha \in [0,1]^n$, if for all $i \in I$ the following inequality holds,*

$$u_i^{\alpha_i}(\tau_i^*, \tau_{-i}^*) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}^*),\ \forall\ \tau_i \in X_i.$$

That is, τ^* is a Nash equilibrium if and only if $\tau_i^* \in BR_i^{\alpha_i}(\tau_{-i}^*)$ for all $i \in I$.

3. Existence of Nash equilibrium

We assume that the payoffs of each player are random variables following a certain distribution. We consider various cases and show the existence of a mixed strategy Nash equilibrium of chance-constrained game for different values of α.

3.1. Payoff with one random variable component

We consider the case where for each player i, $i \in I$, there exists an action profile $a^i \in A$ such that $r_i(a^i)$ is a random variable whose inverse cumulative distribution function (CDF) $F^{-1}_{r_i(a^i)}(\cdot)$ exists. For all $a \in A \setminus \{a^i\}$, $r_i(a) \in \mathbb{R}$. For a given strategy profile τ, we have from (2.1)

$$r_i(\tau) = r_i(a^i) \prod_{j=1}^{n} \tau_j(a^i_j) + c_i,$$

where $r_i(a^i) \prod_{j=1}^{n} \tau_j(a^i_j)$ is a random variable and the constant c_i is given below,

$$c_i = \sum_{a \in A; a \neq a^i} \prod_{j=1}^{n} \tau_j(a_j) r_i(a).$$

For a given $\tau \in X$ such that $\tau_j(a^i_j) > 0$ for all $j \in I$, we have from (2.2),

$$\begin{aligned} u_i^{\alpha_i}(\tau) &= \sup \left\{ u \middle| P\left(r_i(a^i) \prod_{j=1}^{n} \tau_j(a^i_j) + c_i \geq u \right) \geq \alpha_i \right\} \\ &= \sup \left\{ u \middle| P\left(r_i(a^i) \leq \frac{u - c_i}{\prod_{j=1}^{n} \tau_j(a^i_j)} \right) \leq 1 - \alpha_i \right\} \\ &= \sup \left\{ u \middle| u \leq c_i + \prod_{j=1}^{n} \tau_j(a^i_j) F^{-1}_{r_i(a^i)}(1 - \alpha_i) \right\}. \end{aligned}$$

That is,

$$u_i^{\alpha_i}(\tau) = \sum_{a \in A; a \neq a^i} \prod_{j=1}^{n} \tau_j(a_j) r_i(a) + \prod_{j=1}^{n} \tau_j(a^i_j) F^{-1}_{r_i(a^i)}(1 - \alpha_i), \; i \in I. \tag{3.1}$$

If $\tau_j(a^i_j) = 0$ for some $j \in I$, then from (2.2) we obtain

$$\begin{aligned} u_i^{\alpha_i}(\tau) &= \sup \left\{ u | P\left(c_i \geq u \right) \geq \alpha_i \right\} \\ &= \sup \left\{ u | c_i \geq u \right\}. \end{aligned}$$

That is,

$$u_i^{\alpha_i}(\tau) = \sum_{a \in A; a \neq a^i} \prod_{j=1}^{n} \tau_j(a_j) r_i(a).$$

So, for all $\tau \in X$ the payoff of player i, $i \in I$, is given by (3.1). We can write (3.1) as,

$$u_i^{\alpha_i}(\tau) = \sum_{a \in A} \prod_{j=1}^{n} \tau_j(a_j) \tilde{r}_i(a), \ i \in I,$$

where,

$$\tilde{r}_i(a) = \begin{cases} F^{-1}_{r_i(a^i)}(1 - \alpha_i), \ \text{if } a = a^i, \\ r_i(a), \ \text{if } a \neq a^i. \end{cases}$$

Hence, the above game is equivalent to a deterministic game with payoff function $\tilde{r}_i : A \to \mathbb{R}$ for player i, $i \in I$. Therefore, for each $\alpha \in [0, 1]^n$ there always exists a mixed strategy Nash equilibrium [2].

3.2. Payoffs following normal distribution

We consider the situation where payoffs of each player are independent normal random variables. We consider two different cases. For each case we show the existence of a mixed strategy Nash equilibrium of corresponding chance-constrained game for different values of α.

3.2.1. Random payoffs for each player corresponding to only one of his action

We consider the case where for each player i, $i \in I$, there exists an action $\bar{a}_i \in A_i$ such that $\{r_i(\bar{a}_i, a_{-i})\}_{a_{-i} \in A_{-i}}$ are independent random variables where for each $a_{-i} \in A_{-i}$, $r_i(\bar{a}_i, a_{-i})$ follows a normal distribution with mean $\mu_i(\bar{a}_i, a_{-i})$ and variance $\sigma_i^2(\bar{a}_i, a_{-i})$. The rest of the payoffs are deterministic. For a given strategy profile $\tau \in X$, we have from (2.1),

$$\begin{aligned} r_i(\tau) &= \tau_i(\bar{a}_i) \left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j(a_j) r_i(\bar{a}_i, a_{-i}) \right) + d_i \\ &= \tau_i(\bar{a}_i) Y_i(\bar{a}_i) + d_i, \end{aligned}$$

where $Y_i(\bar{a}_i) = \left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j(a_j) r_i(\bar{a}_i, a_{-i}) \right)$ is a random variable and the constant d_i is given below,

$$d_i = \sum_{a_i \in A_i; a_i \neq \bar{a}_i} \tau_i(a_i) \left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j(a_j) r_i(a_i, a_{-i}) \right).$$

It is well known that a linear combination of independent normal random variables follows a normal distribution. That is, $Y_i(\bar{a}_i)$ follows a normal distribution with mean $\mu_i(\bar{a}_i) = \sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j(a_j) \mu_i(\bar{a}_i, a_{-i})$ and variance $\sigma_i^2(\bar{a}_i) = \sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j^2(a_j) \sigma_i^2(\bar{a}_i, a_{-i})$. Hence, $Z_i = \frac{Y_i(\bar{a}_i) - \mu_i(\bar{a}_i)}{\sigma_i(\bar{a}_i)}$ follows a standard normal distribution. For a given strategy profile $\tau \in X$ such that $\tau_i(\bar{a}_i) > 0$, we have from (2.2),

$$\begin{aligned}
u_i^{\alpha_i}(\tau) &= \sup \left\{ u | P\left(\tau_i(\bar{a}_i) Y_i(\bar{a}_i) + d_i \geq u\right) \geq \alpha_i \right\} \\
&= \sup \left\{ u | P\left(Z_i \leq \frac{\frac{u - d_i}{\tau_i(\bar{a}_i)} - \mu_i(\bar{a}_i)}{\sigma_i(\bar{a}_i)} \right) \leq 1 - \alpha_i \right\} \\
&= \sup \left\{ u | u \leq d_i + \tau_i(\bar{a}_i) \left(\mu_i(\bar{a}_i) + \sigma_i(\bar{a}_i) F_{Z_i}^{-1}(1 - \alpha_i) \right) \right\} \\
&= d_i + \tau_i(\bar{a}_i) \left(\mu_i(\bar{a}_i) + \sigma_i(\bar{a}_i) F_{Z_i}^{-1}(1 - \alpha_i) \right).
\end{aligned}$$

If $\tau_i(\bar{a}_i) = 0$, then from (2.2), $u_i(\tau) = d_i$. That is, the payoff of player i, $i \in I$, for a given $\tau \in X$ is given by,

$$\begin{aligned}
u_i^{\alpha_i}(\tau) = \sum_{a_i \in A_i; a_i \neq \bar{a}_i} \tau_i(a_i) &\left(\sum_{a_{-i} \in A_{-i}} \prod_{j=1; j \neq i}^{n} \tau_j(a_j) r_i(a_i, a_{-i}) \right) \\
&+ \tau_i(\bar{a}_i) \left(\mu_i(\bar{a}_i) + \sigma_i(\bar{a}_i) F_{Z_i}^{-1}(1 - \alpha_i) \right). \qquad (3.2)
\end{aligned}$$

Theorem 3.1. *Consider an n-player game where each player has finite number of actions. If for each player i, $i \in I$, there exists an action $\bar{a}_i \in A_i$ such that the payoffs $\{r_i(\bar{a}_i, a_{-i})\}_{a_{-i} \in A_{-i}}$ are independent random variables, where $r_i(\bar{a}_i, a_{-i})$ follows a normal distribution with mean $\mu_i(\bar{a}_i, a_{-i})$ and variance $\sigma_i^2(\bar{a}_i, a_{-i})$, and all other payoffs are deterministic, then there exists a mixed strategy Nash equilibrium for all $\alpha \in [0,1]^n$.*

Proof. Let $\mathcal{P}(X)$ be a power set of X. Define a set valued map $G : X \to \mathcal{P}(X)$ such that $G(\tau) = \prod_{i=1}^{n} BR_i^{\alpha_i}(\tau_{-i})$. A strategy profile $\tau \in X$ is said to be a fixed point of a set valued map G if $\tau \in G(\tau)$. It is easy to see that a fixed point of G is a Nash equilibrium. So, it is sufficient to show that G has a fixed point. In order to show that G has a fixed point, we show that G satisfies all the following conditions of Kakutani fixed point theorem [17]:

1. X is a non-empty, convex and compact subset of a finite dimensional Euclidean space.

2. $G(\tau)$ is non-empty and convex for all $\tau \in X$.

3. $G(\cdot)$ has closed graph: If $(\tau_n, \bar{\tau}_n) \to (\tau, \bar{\tau})$ with $\bar{\tau}_n \in G(\tau_n)$ for all n then $\bar{\tau} \in G(\tau)$.

Condition 1 holds from the definition of X. Fix $\alpha \in [0,1]^n$. For fixed τ_{-i}, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a continuous function of τ_i from (3.2). So, $BR_i^{\alpha_i}(\tau_{-i})$ is non-empty for each $i \in I$ because a continuous function $u_i^{\alpha_i}(\cdot, \tau_{-i})$ over a compact set X_i always attains maxima. Hence, $G(\tau)$ is non-empty for all $\tau \in X$. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (3.2) is a linear function of τ_i. Hence, $G(\tau)$ is a convex set for all $\tau \in X$. Now, we prove that $G(\cdot)$ is a closed graph. Assume that $G(\cdot)$ is not a closed graph, i.e., there is a sequence $(\tau^n, \bar{\tau}^n) \to (\tau, \bar{\tau})$ with $\bar{\tau}^n \in G(\tau^n)$ for all n, but $\bar{\tau} \notin G(\tau)$. In this case $\bar{\tau}_i \notin BR_i(\tau_{-i})$ for some $i \in I$. Then, there is an $\epsilon > 0$ and a $\tilde{\tau}_i$ such that

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) > u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) + 3\epsilon. \tag{3.3}$$

Since $u_i^{\alpha_i}(\cdot)$ is a continuous function of τ from (3.2), $u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) \to u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i})$. Then there is an integer N_1 such that

$$u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) < u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) + \epsilon, \ \forall\, n \geq N_1. \tag{3.4}$$

From (3.3) and (3.4), we have

$$u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) < u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) - 2\epsilon, \ \forall\, n \geq N_1. \tag{3.5}$$

Similarly, $u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) \to u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i})$. So, there is an integer N_2 such that

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}) < u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) + \epsilon, \ \forall \ n \geq N_2. \tag{3.6}$$

Let $N = \max\{N_1, N_2\}$. Then, from (3.5) and (3.6), we have

$$u_i^{\alpha_i}(\tilde{\tau}_i, \tau_{-i}^n) > u_i^{\alpha_i}(\bar{\tau}_i^n, \tau_{-i}^n) + \epsilon, \ \forall \ n \geq N.$$

That is, $\tilde{\tau}_i$ performs better than $\bar{\tau}_i^n$ against τ_{-i}^n for all $n \geq N$ which contradicts $\bar{\tau}_i^n \in BR_i^{\alpha_i}(\tau_{-i}^n)$ for all n. Hence, $G(\cdot)$ is a closed graph. That is, the set valued map $G(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem. Hence, $G(\cdot)$ has a fixed point τ^*, i.e., $\tau^* \in G(\tau^*)$. Such τ^* is a Nash equilibrium of the game. The $\alpha \in [0,1]^n$ is arbitrary, so, there always exists a mixed strategy Nash equilibrium for all $\alpha \in [0,1]^n$.

3.2.2. Random payoffs for each player corresponding to all action profiles

We consider the case where all the payoffs of each player are independent normal random variables. We assume that for each $i \in I$, $\{r_i(a)\}_{a \in A}$ are independent random variables, where $r_i(a)$ follows a normal distribution with mean $\mu_i(a)$ and variance $\sigma_i^2(a)$. So, for $\tau \in X$, $r_i(\tau)$ follows a normal distribution with mean $\mu_i = \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\mu_i(a)$ and variance $\sigma_i^2 = \sum_{a \in A} \prod_{j=1}^n \tau_j^2(a_j)\sigma_i^2(a)$. So, $Z_i = \frac{r_i(\tau) - \mu_i}{\sigma_i}$ follows a standard normal distribution. Similarly to the previous case, the payoff of player i, $i \in I$, for a given $\tau \in X$ is given by,

$$u_i^{\alpha_i}(\tau) = \sum_{a \in A} \prod_{j=1}^n \tau_j(a_j)\mu_i(a) + \left(\sum_{a \in A} \prod_{j=1}^n \tau_j^2(a_j)\sigma_i^2(a) \right)^{\frac{1}{2}} F_{Z_i}^{-1}(1 - \alpha_i). \tag{3.7}$$

For fixed τ_{-i}, the first term of $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined in (3.7) is a linear function of τ_i. If $\alpha_i \in [0.5, 1]$, then the second term of $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i because $F_{Z_i}^{-1}(1 - \alpha_i) \leq 0$ for all $\alpha_i \in [0.5, 1]$. So, we can say that for each $i \in I$, $u_i(\cdot, \tau_{-i})$ is a concave function of τ_i for all $\alpha_i \in [0.5, 1]$. Then, for each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set for all $\alpha_i \in [0.5, 1]$. For each $i \in I$, $u_i^{\alpha_i}(\cdot)$ given in (3.7) is also a continuous function of τ. We have the following result in this case.

Theorem 3.2. *Consider an n-player game where each player has finite number of actions. If for each player i, $i \in I$, $\{r_i(a)\}_{a\in A}$ are independent random variables, where $r_i(a)$ follows a normal distribution with mean $\mu_i(a)$ and variance $\sigma_i^2(a)$, then there exists a mixed strategy Nash equilibrium for all $\alpha \in [0.5, 1]^n$.*

PROOF. Using the fact that $BR_i^{\alpha_i}(\tau_{-i})$, $i \in I$, is a convex set for all $\alpha_i \in [0.5, 1]$ and $u_i^{\alpha_i}(\cdot)$ is a continuous function of τ, the proof follows from similar arguments as those given in Theorem 3.1.

3.3. Payoffs following multivariate elliptical distributions

We assume that a vector of payoffs $(r_i(a))_{a\in A}$ of each player i, $i \in I$, follows a multivariate elliptically symmetric distribution with parameters μ_i and Σ_i, where vector $\mu_i = (\mu_i(a))_{a\in A}$ represents a location parameter and Σ_i is a scale matrix. We assume Σ_i to be a positive definite matrix. Then, all linear combinations of the components of payoff vector follow a univariate elliptically symmetric distribution [18]. Let $\eta = (\eta(a))_{a\in A}$ be a vector, where $\eta(a) = \prod_{j=1}^n \tau_j(a_j)$. Then for $\tau \in X$, $r_i(\tau)$, $i \in I$, follows a univariate elliptically symmetric distribution with parameters $\eta^T\mu_i$ and $\eta^T\Sigma_i\eta$. Since Σ_i is a positive definite matrix, $\sqrt{\eta^T\Sigma_i\eta}$ will be a norm and it is denoted by $||\eta||_{\Sigma_i}$. For each $i \in I$, $Z_i = \frac{r_i(\tau)-\eta^T\mu_i}{||\eta||_{\Sigma_i}}$ follows a univariate spherically symmetric distribution with parameters 0 and 1 [18]. From (2.2), for a given $\tau \in X$, we obtain

$$\begin{aligned}
u_i^{\alpha_i}(\tau) &= \sup\{u|P(r_i(\tau) \geq u) \geq \alpha_i\} \\
&= \sup\left\{u|P\left(\frac{r_i(\tau)-\eta^T\mu_i}{||\eta||_{\Sigma_i}} \leq \frac{u-\eta^T\mu_i}{||\eta||_{\Sigma_i}}\right) \leq 1-\alpha_i\right\} \\
&= \sup\left\{u|u \leq \eta^T\mu_i + ||\eta||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i)\right\} \\
&= \eta^T\mu_i + ||\eta||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i).
\end{aligned}$$

That is,

$$u_i^{\alpha_i}(\tau) = \sum_{a\in A}\prod_{j=1}^n \tau_j(a_j)\mu_i(a) + ||\eta||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i),\ i \in I. \tag{3.8}$$

Fix τ_{-i}, then we can rewrite (3.8) as

$$u_i^{\alpha_i}(\tau_i, \tau_{-i}) = h(\tau_i) + ||\eta||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i),\ i \in I, \tag{3.9}$$

where $h(\cdot)$ is a linear function of τ_i. We know that the quantile function $F_{Z_i}^{-1}(1-\alpha_i) \leq 0$ for all $\alpha_i \in (0.5, 1]$, if Z_i has strictly positive density then $F_{Z_i}^{-1}(1-\alpha_i) \leq 0$ for all $\alpha_i \in [0.5, 1]$ (see [19]).

Lemma 3.3. *$u_i^{\alpha_i}(\cdot, \tau_{-i})$, $i \in I$, is a concave function of τ_i for all $\alpha_i \in (0.5, 1]$.*

Proof. Fix $\alpha_i \in (0.5, 1]$ and $i \in I$. Let $\tau_i^1, \tau_i^2 \in X_i$. Let $\eta^k = (\eta^k(a))_{a \in A}$, where $\eta^k(a) = \tau_i^k(a_i) \prod_{j=1;j\neq i}^n \tau_j(a_j)$, $k = 1, 2$. For $\lambda \in [0, 1]$, $\lambda\eta^1 + (1-\lambda)\eta^2 = \big(\lambda\eta^1(a) + (1-\lambda)\eta^2(a)\big)_{a\in A}$, where,

$$\lambda\eta^1(a) + (1-\lambda)\eta^2(a) = \big(\lambda\tau_i^1(a_i) + (1-\lambda)\tau_i^2(a_i)\big) \prod_{j=1;j\neq i}^{n} \tau_j(a_j).$$

Then, from (3.9) we have for all $\lambda \in [0, 1]$,

$$\begin{aligned} u_i^{\alpha_i}(\lambda\tau_i^1 + (1-\lambda)\tau_i^2, \tau_{-i}) &= h(\lambda\tau_i^1 + (1-\lambda)\tau_i^2) + ||\lambda\eta^1 + (1-\lambda)\eta^2||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i) \\ &\geq \lambda\left(h(\tau_i^1) + ||\eta^1||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i)\right) \\ &\qquad + (1-\lambda)\left(h(\tau_i^2) + ||\eta^2||_{\Sigma_i} F_{Z_i}^{-1}(1-\alpha_i)\right) \\ &= \lambda u_i^{\alpha_i}(\tau_i^1, \tau_{-i}) + (1-\lambda) u_i^{\alpha_i}(\tau_i^2, \tau_{-i}). \end{aligned}$$

So, we can say that for each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of τ_i for all $\alpha_i \in (0.5, 1]$.

Remark 3.4. *If payoff vector $(r_i(a))_{a\in A}$, $i \in I$, has strictly positive density then Lemma 3.3 holds for all $\alpha_i \in [0.5, 1]$.*

Theorem 3.5. *Consider an n-player game where each player has finite number of actions. If a payoff vector $(r_i(a))_{a\in A}$ of player i, $i \in I$, follows a multivariate elliptically symmetric distribution with location parameter $\mu_i = (\mu_i(a))_{a\in A}$ and scale matrix Σ_i which is positive definite, then there exists a mixed strategy Nash equilibrium for all $\alpha \in (0.5, 1]^n$.*

Proof. For each $i \in I$, $BR_i^{\alpha_i}(\tau_{-i})$ is a convex set for all $\alpha_i \in (0.5, 1]$ from Lemma 3.3. From (3.8), $u_i(\cdot)$ is a continuous function of τ. Then, the proof follows using similar arguments as those given in Theorem 3.1.

Remark 3.6. *For each $i \in I$, if payoff vector $(r_i(a))_{a\in A}$ has strictly positive density then Theorem 3.5 holds for all $\alpha \in [0.5, 1]^n$.*

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