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DISTRIBUTIONALLY ROBUST STOCHASTIC KNAPSACK PROBLEM

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Distributionally robust stochastic knapsack problem

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1 Deterministic problem

We consider the classical quadratic knapsack problem consisting of the decision to include or not each of a list of n item in a bag able to carry a certain maximum weight. An item i is associated with a pair (r_i, w_i) describing the added reward (in terms of utility of the knapsack's content) and weight for including it. This problem takes the form of the following optimization problem

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \mathbf{x}^T \mathbf{R} \mathbf{x} && \text{(1a)}\\
\text{subject to} \quad & \mathbf{w}^T \mathbf{x} \leq d && \text{(1b)}\\
& x_i \in \{0,1\}\ , \forall\, i \in \{1,2,...,n\}\ , && \text{(1c)}
\end{aligned}$$

where $\mathbf{x}$ is a vector of binary values indicating whether each item is included in the knapsack or not, $\mathbf{w} \in \Re^n$ is the vector of weights, and $\mathbf{R} \in \Re^{n\times n}$ is a matrix which (i,j) term describes the linear contribution to reward of holding both items i and j. This allows to model complementarity or substitution between items.

2 Stochastic problem

It is often the case that at the time of making the knapsack decision either the reward parameters or the weights parameter (or both) are not exactly known. In that case, one has the option to represent is knowledge of these parameters through describing a measurable space of outcomes $(\Omega, \mathcal{F})$ and a probability measure F on this space. The knapsack problem thus becomes a stochastic problem where $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{w}}$ must now respectively be considered as a random matrix and a random vector. Specifically, $\tilde{\mathbf{R}} : \Omega \rightarrow \Re^{n\times n}$ and $\tilde{\mathbf{w}} : \Omega \rightarrow \Re^n$. In this context, it is natural to formulate the following stochastic program:

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \mathbb{E}_F[u(\mathbf{x}^T \tilde{\mathbf{R}} \mathbf{x})] && \text{(2a)}\\
\text{subject to} \quad & \mathbb{P}_F(\tilde{\mathbf{w}}^T \mathbf{x} \leq d) \geq 1-\eta && \text{(2b)}\\
& x_i \in \{0,1\}\ , \forall\, i \in \{1,2,...,n\}\ , && \text{(2c)}
\end{aligned}$$

for some utility function $u(\cdot)$ and some $0 < \eta < 1$.

3 Distributionally robust problem

Under many circumstances, the assumption of full knowledge of the distribution F fails. For this reason, it can be necessary to consider that the only knowledge we have of a distribution is that it is part of some uncertainty set $\mathcal{D}$. Following a robust approach, in this context we will be interested in choosing items for out knapsack so that the value of the knapsack, as measured by the stochastic program, as best worst-case guarantees under the choice of a distribution in this uncertainty set. By exploiting the Lagrangian of the stochastic program, this can be represented mathematically as

$$\underset{\mathbf{x}}{\text{maximize}} \quad \inf_{F \in \mathcal{D}} \quad \text{SP}(\mathbf{x}, F) \ , \tag{3}$$

where $\text{SP}(\mathbf{x}, F)$ refers to the objective function of this problem that is augmented with feasibility verification: *i.e.*,

$$\text{SP}(\mathbf{x}; F) = \begin{cases} \mathbb{E}_F[u(\mathbf{x}^T \tilde{\mathbf{R}} x)] & \text{if } \mathbb{P}_F(\tilde{\mathbf{w}}^T x \leq d) \geq 1 - \eta \\ -\infty & \text{otherwise} \end{cases} .$$

Lemma 1. *Problem* (3) *is equivalent to*

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \inf_{F \in \mathcal{D}} \mathbb{E}_F[u(\mathbf{x}^T \tilde{\mathbf{R}} \mathbf{x})] & \text{(4a)} \\ \text{subject to} \quad & \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}^T \mathbf{x} \leq d) \geq 1 - \eta & \text{(4b)} \\ & x_i \in \{0, 1\} \ , \forall \, i \in \{1, 2, ..., n\} \ . & \text{(4c)} \end{aligned}$$

Proof. This is simply due to the fact that in order for $\inf_{F \in \mathcal{D}} \text{SP}(\mathbf{x}, F)$ to be finite valued, $\mathbf{x}$ must satisfy constraint 5b. ☐

4 Reduction of robust problem to deterministic form

Definition 1. *Without loss of generality, let* ξ *be a random vector in* $\Re^m$ *from which* $\tilde{\mathbf{R}}$ *and* $\tilde{\mathbf{w}}$ *depend linearly.*

$$\tilde{\mathbf{R}} = \sum_{i=1}^{m} \mathbf{A}_i^{\tilde{\mathbf{R}}} \xi_i \qquad\qquad \tilde{\mathbf{w}} = \mathbf{A}^{\tilde{\mathbf{w}}} \xi \ .$$

Assumption 1. *The utility function* $u(\cdot)$ *is piecewise linear, increasing and concave. In other words, it can be represented in the form:*

$$u(y) = \min_{i \in \{1,2,\ldots,K\}} a_i y + b_i \ ,$$

where $\mathbf{a} \in \Re^K$ *and* $\mathbf{a} \geq 0$.

Assumption 2. *The distributional uncertainty set accounts for information about the support $\mathcal{S}$, mean μ, and an upper bound Σ on the covariance matrix of the random vector ξ*

$$\mathcal{D}(\mathcal{S},\mu,\Sigma)=\left\{F\;\middle|\;\begin{array}{l}\mathbb{P}(x\in\mathcal{S})=1\\ \mathbb{E}_F[\xi]=\mu\\ \mathbb{E}_F[(\xi-\mu)(\xi-\mu)^T]\preceq\Sigma\end{array}\right\}.$$

Theorem 1. *Under assumptions 1 and 2, and given that $0<\eta<1$, then problem* (3) *is equivalent to the following deterministic problem*

$$\underset{\mathbf{x},t,\mathbf{q},\mathbf{Q}}{\text{maximize}}\quad t-\mu^T\mathbf{q}-(\Sigma+\mu\mu^T)\bullet\mathbf{Q} \tag{5a}$$

$$\text{subject to}\quad t\leq\sum_{j=1}^{m}a_i\xi_j\mathbf{x}^T\mathbf{A}_j^{\tilde{\mathbf{R}}}\mathbf{x}+b_i+\xi^T\mathbf{q}+\xi^T\mathbf{Q}\xi\ ,\ \forall\,\xi\in\mathcal{S}\ \forall\,i=\{1,2,...,K\} \tag{5b}$$

$$\mathbf{Q}\succeq 0 \tag{5c}$$

$$\mu^T\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x}+\sqrt{\frac{1-\eta}{\eta}}\|\Sigma^{1/2}\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x}\|_2\leq d \tag{5d}$$

$$x_i\in\{0,1\}\ ,\ \forall i\in\{1,2,...,n\}\ . \tag{5e}$$

Proof. The proof relies here on applying the theory presented in [Delage & Ye, 2010] to convert the distributionally robust objective into its deterministic equivalent. In a second step, one applies the theory presented in [Calafiore & El Ghaoui, 2006] to convert the distributionally robust chance constraint in its own deterministic equivalent. □

Corollary 1. *Given that the support of F is ellipsoidal, $\mathcal{S}=\{\xi|\xi^T\xi\leq 1\}$, this problem further reduces to the following problem*

$$\underset{\mathbf{x},t,\mathbf{q},\mathbf{Q},\mathbf{v},\mathbf{z},\tau}{\text{maximize}}\quad t-\mu^T\mathbf{q}-(\Sigma+\mu\mu^T)\bullet\mathbf{Q} \tag{6a}$$

$$\text{subject to}\quad \begin{bmatrix}\mathbf{Q} & \frac{\mathbf{q}+a_i\mathbf{v}}{2}\\ \frac{\mathbf{q}^T+a_i\mathbf{v}^T}{2} & b_i-t\end{bmatrix}\succeq-\tau_k\begin{bmatrix}\mathbf{I} & 0\\ 0 & -1\end{bmatrix}\ ,\ \forall\,i=\{1,2,...,K\} \tag{6b}$$

$$v_j=\mathbf{A}_j^{\tilde{\mathbf{R}}}\bullet(\mathbf{x}\mathbf{x}^T)\ ,\ \forall\,j\in\{1,2,...,m\} \tag{6c}$$

$$\mathbf{Q}\succeq 0 \tag{6d}$$

$$\begin{bmatrix}0 & \Sigma^{1/2}\mathbf{z}\\ \mathbf{z}^T\Sigma^{1/2} & 0\end{bmatrix}\succeq\sqrt{\frac{\eta}{1-\eta}}(\mu^T\mathbf{z}-d)\mathbf{I} \tag{6e}$$

$$\mathbf{z}=\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x} \tag{6f}$$

$$x_i\in\{0,1\}\ ,\ \forall i\in\{1,2,...,n\}\ , \tag{6g}$$

which, if we disregard constraint (6c)*, is a semi-definite program with binary variables.*

Proof. Here, the proof relies on the S-Lemma and on a well known equivalence relation for second order cone constraints to reformulate constraint (5b) and 5d respectively as linear matrix inequalities. □

5 Semidefinite Relaxation

We apply semidefinite relaxation of the binary constraints which makes use of the decision matrix $\mathbf{X}$.

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}\mathbf{x}^T & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix}$$

Problem (4) can therefore be rewritten as:

$$\begin{aligned}
\underset{\mathbf{x},t,\mathbf{q},\mathbf{Q},\mathbf{v},\mathbf{z},\tau}{\text{maximize}} \quad & t - \mu^T\mathbf{q} - (\Sigma + \mu\mu^T) \bullet \mathbf{Q} && \text{(7a)} \\
\text{subject to} \quad & \begin{bmatrix} \mathbf{Q} & \frac{\mathbf{q}+a_i\mathbf{v}}{2} \\ \frac{\mathbf{q}^T+a_i\mathbf{v}^T}{2} & b_i - t \end{bmatrix} \succeq -\tau_k \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix} \ , \forall i = \{1, 2, ..., K\} && \text{(7b)} \\
& v_j = \bar{\mathbf{A}}_j^{\tilde{\mathbf{R}}} \bullet \mathbf{X} \ , \forall j \in \{1, 2, ..., m\} && \text{(7c)} \\
& \mathbf{Q} \succeq 0 && \text{(7d)} \\
& \begin{bmatrix} 0 & \Sigma^{1/2}\mathbf{z} \\ \mathbf{z}^T\Sigma^{1/2} & 0 \end{bmatrix} \succeq \sqrt{\frac{\eta}{1-\eta}}(\mu^T\mathbf{z} - d)\mathbf{I} && \text{(7e)} \\
& z_i = \bar{\mathbf{A}}_i^{\tilde{\mathbf{w}}} \bullet \mathbf{X} && \text{(7f)} \\
& \mathbf{X} \in \mathcal{X} \ , && \text{(7g)}
\end{aligned}$$

where $\mathcal{X}$ represents a set of constraints that serve the purpose of tightening the relaxation, and where

$$\bar{\mathbf{A}}_j^{\tilde{\mathbf{R}}} = \begin{bmatrix} \mathbf{A}_j^{\tilde{\mathbf{R}}} & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{\mathbf{A}}_i^{\tilde{\mathbf{w}}} = \begin{bmatrix} 0 & 0.5[\mathbf{A}^{\tilde{\mathbf{w}}}]_i \\ 0.5[\mathbf{A}^{\tilde{\mathbf{w}}}]_i^T & 1 \end{bmatrix} ,$$

with the notation $[\cdot]_i$ used to refer to an operator that extracts the i-th column of a matrix.

6 Robust problem for multidimensional knapsack problem

Here we consider stochastic multidimensional knapsack problems. The problem can be formulated as below:

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \mathbb{E}_F[u(\mathbf{x}^T\tilde{\mathbf{R}}\mathbf{x})] && \text{(8a)} \\
\text{subject to} \quad & \mathbb{P}_F(\tilde{\mathbf{w}}_j^T\mathbf{x} \le d_j, j = 1, \ldots, M) \ge 1 - \eta && \text{(8b)} \\
& x_i \in \{0, 1\} \ , \forall i \in \{1, 2, ..., n\} \ , && \text{(8c)}
\end{aligned}$$

for some utility function $u(\cdot)$ and some $0 < \eta < 1$.

And the corresponding robust problem for the multiple knapsack problem is

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \inf_{F \in \mathcal{D}} \mathbb{E}_F[u(\mathbf{x}^T \tilde{\mathbf{R}} \mathbf{x})] & \text{(9a)} \\
\text{subject to} \quad & \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta & \text{(9b)} \\
& x_i \in \{0, 1\} \ , \forall \, i \in \{1, 2, ..., n\} \ . & \text{(9c)}
\end{aligned}$$

Definition 2. *Without loss of generality, let* $\xi_j, j = 1, \ldots, M$ *be a random vector in* $\Re^m$ *from which* $\tilde{\mathbf{R}}$ *and* $\tilde{\mathbf{w}}$ *depend linearly.*

$$\tilde{\mathbf{R}} = \sum_{j=1}^{M} \sum_{i=1}^{m} \mathbf{A}_{ji}^{\tilde{\mathbf{R}}} \xi_{ji} \qquad\qquad \tilde{\mathbf{w}}_j = \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \xi_j, j = 1, \ldots, M \ .$$

Assumption 3. *The distributional uncertainty set accounts for information about the support* $\mathcal{S}$*, mean* μ_j*, and an upper bound* Σ_j *on the covariance matrix of the random vector* $\xi_j, j = 1, \ldots, M$

$$\mathcal{D}(\mathcal{S}, \mu_j, \Sigma_j) = \left\{ F \left| \begin{array}{l} \mathbb{P}(x \in \mathcal{S}) = 1 \\ \mathbb{E}_F[\xi_j] = \mu_j \\ \mathbb{E}_F[(\xi_j - \mu_j)(\xi_j - \mu_j)^T] \preceq \Sigma_j \end{array} \right. \right\} .$$

Furthermore, the random vectors ξ_i *and* ξ_j *are independent when* $i \neq j$*. The support of* F *is ellipsoidal,* $\mathcal{S} = \{\xi | \xi^T \xi \leq 1\}$*.*

Theorem 2. *Under assumptions 1 and 3, and given that* $0 < \eta < 1$*, then problem (9) is equivalent to the following problem*

$$\begin{aligned}
\underset{\mathbf{x}, t, \mathbf{q}_j, \mathbf{Q}}{\text{maximize}} \quad & t - \mu^T \mathbf{q} - (\Sigma + \mu\mu^T) \bullet \mathbf{Q} & \text{(10a)} \\
\text{subject to} \quad & \begin{bmatrix} \mathbf{Q} & \frac{\mathbf{q} + a_k \mathbf{v}}{2} \\ \frac{\mathbf{q}^T + a_k \mathbf{v}^T}{2} & b_k - t \end{bmatrix} \succeq -\tau_i \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix} \ , \forall \, k = \{1, 2, ..., K\} & \text{(10b)} \\
& v_{(j-1)*M+i} = \mathbf{A}_{ji}^{\tilde{\mathbf{R}}} \bullet (\mathbf{x}\mathbf{x}^T) \ , \forall \, j \in \{1, 2, ..., M\} \forall \, i \in \{1, 2, ..., m\} & \text{(10c)} \\
& \mathbf{Q} \succeq 0 & \text{(10d)} \\
& \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta & \text{(10e)} \\
& x_i \in \{0, 1\} \ , \forall \, i \in \{1, 2, ..., n\} & \text{(10f)}
\end{aligned}$$

where $\mathbf{q} = [\mathbf{q}_1; \ldots; \mathbf{q}_j; \ldots; \mathbf{q}_M]$.

6.1 SDP approximation

Correspondingly, we have a semidefinite relaxation of the multidimensional knapsack problems:

$$
\begin{aligned}
&\underset{\mathbf{x},t,\mathbf{q}_j,\mathbf{Q}}{\text{maximize}} && t - \mu^T\mathbf{q} - (\Sigma + \mu\mu^T) \bullet \mathbf{Q} && \text{(11a)}\\
&\text{subject to} && \begin{bmatrix} \mathbf{Q} & \frac{\mathbf{q}+a_k\mathbf{v}}{2} \\ \frac{\mathbf{q}^T+a_k\mathbf{v}^T}{2} & b_k - t \end{bmatrix} \succeq -\tau_i \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix} \ , \forall\, k = \{1,2,...,K\} && \text{(11b)}\\
& && v_{(j-1)*M+i} = \mathbf{A}^{\tilde{\mathbf{R}}}_{ji} \bullet X \ , \forall\, j \in \{1,2,...,M\} \forall\, i \in \{1,2,...,m\} && \text{(11c)}\\
&(DRSKP - SDP) && \mathbf{Q} \succeq 0 && \text{(11d)}\\
& && \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T\mathbf{x} \le d_j, j = 1, \ldots, M) \ge 1 - \eta && \text{(11e)}\\
& && X_{ii} = x_i, i = 1, \ldots, n && \text{(11f)}\\
& && \begin{bmatrix} 1 & x^T \\ x & \mathbf{X} \end{bmatrix} \succeq 0 && \text{(11g)}\\
& && x_i \in \{0,1\} \ , \forall\, i \in \{1,2,...,n\} && \text{(11h)}
\end{aligned}
$$

Apart from the constraint (11e), DRSKP-SDP is an SDP problem. Thus in this subsection, we investigate the distributionally robust joint chance constraint. First of all, we review two existing approximations.

6.1.1 Bonferroni

A popular approximation for joint chance-constrained problems is based on Bonferroni's inequality, which decomposes the joint constraint into M individual constraints. When $\sum_{j=1}^{M} \eta_k = \eta$, we have

$$\mathbb{P}_F(\tilde{\mathbf{w}}_j^T\mathbf{x} \le d_j) \ge 1 - \eta_j, j = 1, \cdots, M \Rightarrow \mathbb{P}_F(\tilde{\mathbf{w}}_j^T\mathbf{x} \le d_j, j = 1, \ldots, M) \ge 1 - \eta$$

It is easy to prove that the optimal values of the Bonferroni approximations is lower bounds of DRSKP-SDP.

6.1.2 Approximation by Zymler, Kuhn and Rustem

In their approximation, they introduce a scaling parameter $\alpha \in \mathcal{A} = \{\alpha \in R^M : \alpha > 0\}$. For any $\alpha \in \mathcal{A}$

$$\inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T\mathbf{x} \le d_j, j = 1, \ldots, M) \ge 1 - \eta \Longleftrightarrow \inf_{F \in \mathcal{D}} \mathbb{P}_F(\max_j\{\alpha_j(\tilde{\mathbf{w}}_j^T\mathbf{x} - d_j)\} \le 0) \ge 1 - \eta$$

Then they consider the distributionnally robust CVaR conservative approximation :

$$\mathcal{Z}(\alpha) = \{\mathbf{x} \in \mathbb{R}^n : \sup_{F \in \mathcal{D}} \text{CVaR}_\eta(\max_j\{\alpha_j(\tilde{\mathbf{w}}_j^T\mathbf{x} - d_j)\} \le 0\}$$

Theorem 3. *For any fixed $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}$, we have*

$$\mathcal{Z}(\alpha) = \left\{ \mathbf{x} \in \mathbb{R}^n : \exists(\beta, \mathcal{M}) \in \mathbb{R} \times \mathbb{S}^{mM+1} : \left| \begin{array}{l} \beta + \frac{1}{\eta}\Omega \bullet \mathcal{M} \leq 0 \\ \mathcal{M} - \begin{pmatrix} 0 & \frac{1}{2}\alpha_j y_j(\mathbf{x}) \\ \frac{1}{2}\alpha_j y_j^T(\mathbf{x}) & -\alpha_j d_j - \beta \end{pmatrix} \succcurlyeq 0, \ \forall j = 1, \ldots, M \\ \mathcal{M} \succcurlyeq 0 \end{array} \right. \right\}$$

Moreover, $\{\mathbf{x} \in \mathbb{R}^n : \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta\} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{F}(\alpha)$.

where $y_j(\mathbf{x}) \in \mathbb{R}^{mM}$ and $y_j(\mathbf{x}) = (\text{zeros}((j-1)m), \mathbf{A_j}^{\tilde{\mathbf{w}}_j T}\mathbf{x}, \text{zeros}((M-j)m))$, where zeros$(k)$ represents k zero elements.

6.2 Approximation of DRSKP-SDP

Theorem 4. *Under assumptions 1 and 3, and given that $0 < \eta < 1$, then problem DRSKP-SDP is equivalent to the following deterministic problem*

$$\begin{aligned}
&\underset{\mathbf{x},t,\mathbf{q},\mathbf{Q},\mathbf{v},\mathbf{z},\tau}{\text{maximize}} && t - \mu^T\mathbf{q} - (\Sigma + \mu\mu^T) \bullet \mathbf{Q} && \text{(12a)}\\
&\text{subject to} && \begin{bmatrix} \mathbf{Q} & \frac{\mathbf{q}+a_k\mathbf{v}}{2} \\ \frac{\mathbf{q}^T + a_k\mathbf{v}^T}{2} & b_k - t \end{bmatrix} \succeq -\tau_i \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix} \ , \forall\, k = \{1, 2, ..., K\} && \text{(12b)}\\
& && v_{(j-1)*M+i} = \mathbf{A}_{ji}^{\tilde{\mathbf{R}}} \bullet X \ , \forall\, j \in \{1, 2, ..., M\} \forall\, i \in \{1, 2, ..., m\} && \text{(12c)}\\
&(DRSKP - SDP) && \mathbf{Q} \succeq 0 && \text{(12d)}\\
& && \mu^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j}\mathbf{x} + \sqrt{\frac{p^{y_j}}{1 - p^{y_j}}} \|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j T}\mathbf{x}\|_2 \leq d_j && \text{(12e)}\\
& && \sum_{j=1}^{M} y_j = 1, \quad y_j \geq 0 && \text{(12f)}\\
& && X_{ii} = x_i, i = 1, \ldots, n && \text{(12g)}\\
& && \begin{bmatrix} 1 & x^T \\ x & \mathbf{X} \end{bmatrix} \succeq 0 && \text{(12h)}\\
& && && \text{(12i)}
\end{aligned}$$

where $p = 1 - \eta$, $\mu = [\mu_1; \ldots; \mu_j; \ldots; \mu_M]$, $\Sigma = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & \ddots & \\ & & & \Sigma_M \end{bmatrix}$ *and* $\xi = [\xi_1; \ldots; \xi_j; \ldots; \xi_M]$.

6.2.1 Sequential approximations

As we take variables $y_j, j = 1, ...M$ as parameters, the problem DRSKP-SDP is an SDP problem. Thus we use a sequential approximation method where we improve the parameters

y during the procedure. The main idea of the approach is to relax the parameter y_j, i.e., increase value of y_j, when j-th chance constraint is active with feasible solution x_0.

Algorithm 1: *Sequential Approximation Procedure*

- **Initialization:** Let $y^0 \in R_+^M$ be scaling parameters, i.e., $\sum_{j=1}^M y^0(j) = 1, y^0 \geq 0$. Set the iteration counter to $t \leftarrow 1$.
- **Update:** Solving DRSKP-SDP with $y = y^t$ and let x^t and f^t denote an optimal solution and the optimal value, respectively. Let $Y(j) = \frac{(d_j - \mu^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \mathbf{x}^t)^2}{(\|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j T} \mathbf{x}^t\|_2)^2}$ and $\tilde{y}(j) = \log_p \frac{Y(j)}{Y(j)+1}$. Residual parameter: $r = y^t - \tilde{y}$. Weight parameter: $w(j) = 1 - r(j), j = 1, ..., M$. Set $y^t \leftarrow \frac{w \sum r}{\sum w} + \tilde{y}$
- **Stopping criterion:** if $\frac{f^t - f^{t-1}}{f^{t-1}} \leq \epsilon$ (where ϵ is a given small tolerance), return y^t and stop. Otherwise, set $t \leftarrow t+1$ and go back to step **Update**.

Remark: For the initial parameter y^0, all its elements are set to be $\frac{1}{M}$.

Lemma 2. *The sequence y^t are scaling parameters, i.e., $\sum_{j=1}^M y^t(j) = 1, y^t \geq 0$.*

Proof. Following the procedure, we have the updated $y^{t+1} = \frac{w \sum r}{\sum w} + \tilde{y}$. As $\mathbf{x}^*$ is a feasible solution of Problem (12) as $y = y^t$, then $Y(j) = \frac{(d_j - \mu^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \mathbf{x}^*)^2}{(\|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j T} \mathbf{x}^*\|_2)^2} \geq \sqrt{\frac{p^{y^t(j)}}{1 - p^{y^t(j)}}}$, so $\tilde{y}(j) = \log_p \frac{Y(j)}{Y(j)+1} \leq y^t(j)$. Thus $r = y^t - \tilde{y} \geq 0$. Moreover $\sum_{j=1}^M y^t(j) = 1, y^t \geq 0$, $r \leq 1$ and $w(j) = 1 - r(j) \geq 0$. As $p \leq 1$ and $\frac{Y(j)}{Y(j)+1} \leq 1$, so $\tilde{y}(j)$. Thus $y^{t+1} = \frac{w \sum r}{\sum w} + \tilde{y} \geq 0$. Furthermore, $\sum_{j=1}^M y^{t+1}(j) = \sum \frac{w \sum r}{\sum w} + \sum \tilde{y} = \sum r + \sum \tilde{y} = \sum_{j=1}^M y^t(j) = 1$. Thus the conclusion follows. □

Theorem 5. *Assume that the problem DRSKP-SDP has a feasible solution with initial parameter y^0. Then, the sequence of objective values $\{f^t\}$ generated by Algorithm 1 is non-decreasing. Moreover, the sequence $\{f^t\}$ converges to a finite limit and f^t is a lower bound of DRSKP-SDP.*

Proof. By Lemma (2), the updated y^{t+1} in Step Update of the algorithm still are scaling parameters. So the optimal solution x^{t+1} in Step Update of the algorithm is a feasible solution of problem (12). Thus the corresponding optimal value f^{t+1} is a lower bound of DRKSP-SDP. Secondly, we prove that x^t is a feasible solution of (12) when $y = y^{t+1}$. In Step Update of the algorithm, as $Y(j) = \frac{(d_j - \mu^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \mathbf{x}^t)^2}{(\|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j T} \mathbf{x}^t\|_2)^2}$ and $\tilde{y}(j) = \log_p \frac{Y(j)}{Y(j)+1}$, it is easy to verify that $y = \tilde{y}$ and $x = x^t$ satisfy the constraint (12e). In the proof of Lemma (2), we have $w, r \geq 0$ and $y^{t+1} = \frac{w \sum r}{\sum w} + \tilde{y}$, so $y^{t+1} \geq \tilde{y}$. Moreover $\sqrt{\frac{p^y}{1-p^y}}$ is a decreasing function on the interval $(0, 1]$. Thus $y = y^{t+1}$ and $x = x^t$ satisfy the constraint (12e). In other words, x^t is a feasible solution of (12) when $y = y^{t+1}$. This guarantees that the sequences of $\{f^t\}$ is monotonically nondecreasing. Furthermore, since the solution sequence $\{x^t\}$ is bounded and the objective function of (12) is continuous, the monotonicity of the objective value sequence implies that $\{f^t\}$ has a finite limit.

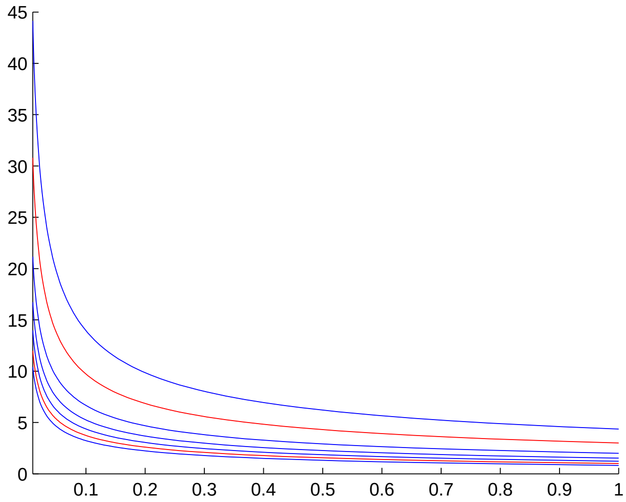
$\square$

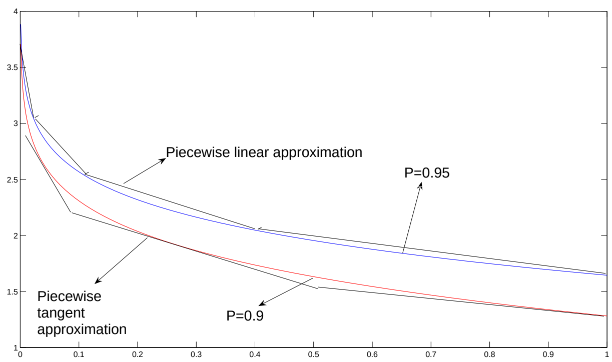
6.2.2 Piecewise tangent approximation

Theorem 6. *Function* $\sqrt{\frac{p^x}{1-p^x}}$ *is convex and decreasing in the region* $(0,1]$*, when* $p \in (0,1)$.

Proof. Since $(\sqrt{\frac{p^x}{1-p^x}})' = \frac{\ln p\sqrt{p^x}}{2(1-p^x)^{\frac{3}{2}}} \leq 0$, when $x \in (0,1], p \in (0,1)$, $\sqrt{\frac{p^x}{1-p^x}}$ is decreasing. $(\sqrt{\frac{p^x}{1-p^x}})'' = \frac{(\ln p)^2}{4}\frac{\sqrt{p^x}(1+2p^x)}{(1-p^x)^{\frac{5}{2}}} \geq 0$, when $x \in (0,1], p \in (0,1)$. So the conclusion claims. $\square$

Here, we draw the function $\sqrt{\frac{p^x}{1-p^x}}$ on the interval $(0,1]$, when $p = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95$ as below.





Then we use piecewise tangent approximation method. The idea to approximate the problem DBSKP is the following: we approximate $\sqrt{\frac{1-\eta^{y_j}}{\eta^{y_j}}}$ with a piecewise tangent approximation of y_j. Afterwards, we get the approximation of DBSKP , which is an SDP problem.

$$
\begin{aligned}
(DRSKP-SDP1)\quad & \underset{\mathbf{x},t,\mathbf{q},\mathbf{Q},\mathbf{v},\mathbf{z},\tau,\tilde{y},\tilde{z}}{\text{maximize}} && t-\mu^T\mathbf{q}-(\Sigma+\mu\mu^T)\bullet\mathbf{Q} && \text{(13a)}\\
& \text{subject to} && \begin{bmatrix}\mathbf{Q} & \frac{\mathbf{q}+a_k\mathbf{v}}{2}\\ \frac{\mathbf{q}^T+a_k\mathbf{v}^T}{2} & b_k-t\end{bmatrix}\succeq-\tau_k\begin{bmatrix}\mathbf{I} & 0\\ 0 & -1\end{bmatrix}\ ,\ \forall k=\{1,2,...,K\} && \text{(13b)}\\
& && v_{(j-1)*M+i}=\mathbf{A}_{ji}^{\tilde{\mathbf{R}}}\bullet X\ ,\ \forall j\in\{1,2,...,m\}\forall i\in\{1,2,...,n\} && \text{(13c)}\\
& && \mathbf{Q}\succeq 0 && \text{(13d)}\\
& && \begin{bmatrix}0 & \Sigma^{1/2}\tilde{\mathbf{z}}_{\mathbf{j}}\\ \tilde{\mathbf{z}}_{\mathbf{j}}{}^T\Sigma^{1/2} & 0\end{bmatrix}\succeq(\mu^T\mathbf{z}_{\mathbf{j}}-d_jt_j)\mathbf{I} && \text{(13e)}\\
& && \mathbf{z}_{\mathbf{j}}=\mathbf{A}_{\mathbf{j}}{}^{\tilde{\mathbf{w}}_j}{}^T\mathbf{x} && \text{(13f)}\\
& && \tilde{\mathbf{z}}_{\mathbf{j}}\geq\hat{a}_l\tilde{\mathbf{y}}_j+\hat{b}_l\mathbf{z}_{\mathbf{j}},\forall l\in\{1,2,...,L\}\forall j\in\{1,2,...,m\} && \text{(13g)}\\
& && \sum_{j=1}^{M}\tilde{\mathbf{y}}_j(i)=\mathbf{z}_{\mathbf{j}}(i),\forall i\in\{1,2,...,n\} && \text{(13h)}\\
& && \sum_{j=1}^{M}y_j=1,\quad y_j\geq 0,\forall j\in\{1,2,...,M\} && \text{(13i)}\\
& && X_{ii}=x_i,i=1,\dots,n && \text{(13j)}\\
& && \begin{bmatrix}1 & x^T\\ x & \mathbf{X}\end{bmatrix}\succeq 0 && \text{(13k)}\\
& && && \text{(13l)}
\end{aligned}
$$

Where

$$
\sqrt{\frac{p^x}{1-p^x}}\approx\max_{l\in\{1,2,...,L\}}\hat{a}_lx+\hat{b}_l\ ,
$$

Theorem 7. *The optimal value of (13) is an upper bound of DRSKP-SDP.*

Proof. By Theorem (6), $\sqrt{\frac{p^x}{1-p^x}}$ is convex and decreasing in the region $(0,1]$. Then by applying the theory presented in [6], the conclusion follows. ☐

6.3 Numerical results part I

We test our formulation SDP relaxations on stochastic knapsack problems (SKP for short) and stochastic multidimentional knapsack problems (SMKP for short) resepctively. For the SKP, instances sizes are represented by two parameters: number of items n and number of random variables m, while there are three parameters for the SMKP: number of items n, number of random variables m and number of joint chance constraints M. We consider four problem sizes, i.e., $(n,m)=(10,5);(10,10);(50,25);(50,50)$. Furthermore, for SKMP, we choose the M to be 5 and 10 respectively.

For the sake of simplicity, we set the utility function $u(y) = y$ and the matrix $\mathbf{R}$ is determinisitc and is generated by MATLAB function "gallery('randcorr',n)*10". The probabilistic capacity constraints are generated with vector means μ_j drawn from the uniform distribution on $[5, 10]$, and the covariance matrix Σ_j generated by MATLAB function "gallery('randcorr',n)*2". The capacity d_j is independently chosen from $[300, 400]$ interval when $n = 10$, while d_j is chosen from $[700, 1000]$ when $n = 50$. The elements of $\mathbf{A_j}^{\tilde{\mathbf{w}}_j}$ are uniformly generated on the interval $[0, 1]$. The Confidence parameter is set to $\alpha = 0.1$.

To mesure the quality of the results of the SDP relaxations designed hereafter by V^{SDP}, we apply randomized rounding method to get a feasible solution whose objective value is a lower bounder designed hereafter by UB. For the SDP approximations of SMKP, we choose three tangent points $z_1 = 0.01$, $z_2 = 0.1$ and $z_3 = 0.4$.

All the considered models are generated using MATLAB environment and solved either by Sedumi [20] with default parameters on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM.

6.4 Numerical results part II

We focus on the semidefinite relaxation of the multidimensional knapsack problems:

$$
(DRSKP-SDP)\quad
\begin{aligned}
\underset{\mathbf{x},t,\mathbf{q},\mathbf{Q},\mathbf{v},\mathbf{z},\tau}{\text{maximize}}\quad & t - \mu^T\mathbf{q} - (\Sigma + \mu\mu^T)\bullet\mathbf{Q} && \text{(14a)}\\
\text{subject to}\quad & \begin{bmatrix} \mathbf{Q} & \frac{\mathbf{q}+a_k\mathbf{v}}{2} \\ \frac{\mathbf{q}^T+a_k\mathbf{v}^T}{2} & b_k - t \end{bmatrix} \succeq -\tau_i \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -1 \end{bmatrix}\ ,\ \forall k = \{1, 2, ..., K\} && \text{(14b)}\\
& v_{(j-1)*M+i} = \mathbf{A}_{ji}^{\tilde{\mathbf{R}}} \bullet X\ ,\ \forall j \in \{1, 2, ..., M\} \forall i \in \{1, 2, ..., m\} && \text{(14c)}\\
& \mathbf{Q} \succeq 0 && \text{(14d)}\\
& \mu^T\mathbf{A}_j^{\tilde{\mathbf{w}}_j}\mathbf{x} + \sqrt{\frac{p^{y_j}}{1-p^{y_j}}}\|\Sigma_j^{1/2}\mathbf{A_j}^{\tilde{\mathbf{w}}_j T}\mathbf{x}\|_2 \le d_j && \text{(14e)}\\
& \sum_{j=1}^{M} y_j = 1, \quad y_j \ge 0 && \text{(14f)}\\
& X_{ii} = x_i, i = 1, \ldots, n && \text{(14g)}\\
& \begin{bmatrix} 1 & x^T \\ x & \mathbf{X} \end{bmatrix} \succeq 0 && \text{(14h)}\\
& && \text{(14i)}
\end{aligned}
$$

We compare the sequential approximation approach with two existing approximations: the Bonferroni approximation and Approximation by Zymler, Kuhn and Rustem. The optimal value of our approximation is denoted by V^I, while the optimal value of the Bonferroni approximation is denoted by V^B and the one of the approximation by Zymler et al is denoted by V^Z. All the optimal values are lower bounds of the DRSKP-SDP. Moreover, we compare them with the piecewise tangent approximation, whose optimal value denoted by V^U

We make the tests on two different size: number of items $n = \{10, 20\}$, the number of random variables $m = \{5, 6\}$ and number of joint chance constraints $M = \{4, 5\}$. For the sake of simplicity, we set the utility function $u(y) = y$ and the matrix $\mathbf{R}$ is deterministic and is generated by MATLAB function "gallery('randcorr',n)*10". The probabilistic capacity constraints are generated with vector means μ_j drawn from the uniform distribution on $[5, 10]$, and the covariance matrix Σ_j generated by MATLAB function "gallery('randcorr',n)*2". The capacity d_j is independently chosen from $[200, 300]$ interval. The elements of $\mathbf{A_j}^{\tilde{\mathbf{w}}_j}$ are uniformly generated on the interval $[0, 1]$.

To measure the quality of the results of the SDP relaxations designed, we apply randomized rounding method to get a feasible solution whose objective value is a lower bounder of the original problem designed hereafter by V^R.

All the considered models are generated using MATLAB environment and solved either by Sedumi [20] with default parameters on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM.

η	V^I	V^Z	V^B	V^U	$\frac{V^U-V^I}{V^I}$	$\frac{V^U-V^Z}{V^Z}$	$\frac{V^U-V^B}{V^B}$	CPU^I	CPU^Z	CPU^B	CPU^U	V^R
1%	46.67	46.65	36.04	46.76	0.19%	0.24%	29.74%	5.60	393.40	0.28	1.31	41.23
2%	54.78	54.74	43.44	54.94	0.29%	0.37%	26.47%	5.80	390.72	0.24	1.13	50.92
3%	59.26	59.20	47.80	59.43	0.29%	0.39%	24.33%	5.45	496.26	0.26	1.09	52.34
4%	62.25	62.18	50.83	62.47	0.35%	0.47%	22.90%	5.62	509.21	0.25	1.10	52.34
5%	64.46	64.38	53.16	64.70	0.37%	0.50%	21.71%	5.83	862.02	0.26	1.14	54.76
6%	66.19	66.10	55.02	66.39	0.30%	0.44%	20.67%	5.86	497.16	0.23	1.05	56.49
7%	67.59	67.50	56.56	67.77	0.27%	0.40%	19.82%	5.74	627.58	0.26	1.17	56.80
8%	68.76	68.66	57.88	68.93	0.25%	0.39%	19.09%	5.75	518.17	0.29	1.15	60.95
9%	69.76	69.66	59.02	69.93	0.24%	0.39%	18.49%	5.82	506.97	0.29	1.21	61.59
10%	70.63	70.53	60.02	70.81	0.25%	0.40%	17.98%	6.21	623.61	0.25	1.53	61.59

Table 1: Computational results of DRSSPP when $n = 10, m = 5, M = 4$

η	V^I	V^Z	V^B	V^U	$\frac{V^U-V^I}{V^I}$	$\frac{V^U-V^Z}{V^Z}$	$\frac{V^U-V^B}{V^B}$	CPU^I	CPU^Z	CPU^B	CPU^U	V^R
1%	60.96	60.94	46.57	61.19	0.38%	0.41%	31.39%	14.70	9311.99	0.61	5.40	50.19
2%	71.61	71.54	55.96	71.83	0.31%	0.41%	28.36%	17.32	8918.62	0.59	4.82	58.36
3%	77.39	77.29	61.50	77.65	0.34%	0.47%	26.26%	17.78	6831.69	0.58	5.18	69.26
4%	81.20	81.08	65.37	81.45	0.31%	0.46%	24.60%	18.52	12461.91	0.63	5.54	72.89
5%	83.95	83.83	68.31	84.23	0.33%	0.48%	23.31%	16.39	9536.62	0.71	5.41	76.61
6%	86.07	85.94	70.66	86.38	0.36%	0.51%	22.25%	17.12	9206.91	0.75	5.41	76.61
7%	87.78	87.64	72.61	88.11	0.38%	0.54%	21.35%	17.18	19749.16	0.72	5.52	76.61
8%	89.18	89.04	74.27	89.51	0.37%	0.53%	20.52%	17.30	9370.42	0.67	5.37	76.61
9%	90.37	90.22	75.70	90.70	0.37%	0.53%	19.82%	16.55	15440.99	0.65	5.22	76.61
10%	91.39	91.24	76.95	91.73	0.37%	0.54%	19.21%	16.98	15970.64	0.68	5.02	77.36

Table 2: Computational results of DRSSPP when $n = 20, m = 6, M = 5$

6.5 Numerical results part IV

In this section, we compare the solution from our proposed distributionally robust approach to the solution of a stochastic programming approach. We test the expected utility performance and chance constraint performance under two different distributions that have the same mean and covariance structure as the assumed. For the stochastic programming approach, we test on two distributions: a normal distribution versus a uniform distribution over an ellipsoid support.

The distributionally robust problem is as follows:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & v^T\mathbf{x} && \text{(15a)} \\ \text{subject to} \quad & \inf_{F\in\mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}^T\mathbf{x} \le d) \ge 1-\eta && \text{(15b)} \\ & 0 \le x_i \le 1 \ , \forall\, i \in \{1, 2, ..., n\} \ , && \text{(15c)} \end{aligned}$$

With assumpitons

Under assumptions 1 and 3, and given that $0 < \eta < 1$, then problem (15) is equivalent to the following problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & v^T\mathbf{x} && (16a)\\ \text{subject to} \quad & \mu^T\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x} + \sqrt{\frac{1-\eta}{\eta}}\|\Sigma^{1/2}\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x}\|_2 \le d && (16b)\\ & 0 \le x_i \le 1 \ , \forall\, i \in \{1,2,...,n\} \ , && (16c) \end{aligned}$$

The stochastic problem is as follows:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & v^T\mathbf{x} && (17a)\\ \text{subject to} \quad & \mathbb{P}_F(\tilde{\mathbf{w}}^T\mathbf{x} \le d) \ge 1-\eta && (17b)\\ & 0 \le x_i \le 1 \ , \forall\, i \in \{1,2,...,n\} \ , && (17c) \end{aligned}$$

When ξ is normal distributed, then problem (17) is equivalent to the following problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & v^T\mathbf{x} && (18a)\\ \text{subject to} \quad & \mu^T\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x} + F^{-1}(1-\eta)\|\Sigma^{1/2}\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x}\|_2 \le d && (18b)\\ & 0 \le x_i \le 1 \ , \forall\, i \in \{1,2,...,n\} \ , && (18c) \end{aligned}$$

where $F^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function.

When ξ is uniformly distributed over an ellipsoid support $\mathcal{S} = \{\xi | \xi^T\xi \le 1\}$, then problem (17) is equivalent to the following problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & v^T\mathbf{x} && (19a)\\ \text{subject to} \quad & \mu^T\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x} + \sqrt{(n+3)(\Psi^{-1}(1-2\eta))}\|\Sigma^{1/2}\mathbf{A}^{\tilde{\mathbf{w}}^T}\mathbf{x}\|_2 \le d && (19b)\\ & 0 \le x_i \le 1 \ , \forall\, i \in \{1,2,...,n\} \ , && (19c) \end{aligned}$$

where $\Psi^{-1}(\cdot)$ is the inverse of the cumulative distribution of a beta(1/2;n/2+1) probability density function.

For the sake of simplicity, we set $\mathbf{A}^{\tilde{\mathbf{w}}}$ to be the identity matrix. The value vector v is generated uniformly from interval $[0, 100]$. The probabilistic capacity constraints are generated with vector means μ drawn from the uniform distribution on $[5, 10]$, and the covariance matrix Σ generated by MATLAB function "gallery('randcorr',n)*2". The capacity d is chosen from $[50, 100]$ interval when $n = 10$, while d is chosen from $[100, 200]$ when $n = 20$.

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n=5				n=10				n=20			
eta	Uniform	Normal	Robust	*eta*	Uniform	Normal	Robust	*eta*	Uniform	Normal	Robust
0.1	0.900	0.890	1.000	0.1	0.900	0.894	1.000	0.1	0.900	0.897	1.000
0.2	0.800	0.782	0.983	0.2	0.800	0.789	0.980	0.2	0.800	0.794	0.979
0.3	0.700	0.684	0.933	0.3	0.700	0.690	0.9344	0.3	0.700	0.695	0.935
0.4	0.600	0.591	0.878	0.4	0.60	0.594	0.883	0.4	0.600	0.597	0.886

Table 3: Computational results

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