RAPPORT DE RECHERCHE

L R I

SOME RESULTS ON (Sigma,p,g)-VALUATION OF CONNECTED GRAPHS

MAHEO M / SACLE J F

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

07/2008

Rapport de Recherche N° 1497

CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

Some results on (Σ, p, g)-valuation of connected graphs .

Maryvonne Mahéo, Jean-François Saclé

U.M.R. 86-23 L.R.I., Bât. 490, Université Paris-Sud 91405 Orsay, France.

Abstract : In this article we study proper valuations $v : E(G) \to \mathbf{N}^*$ on the edges of a graph G, such that the sums of the values taken on the edges incident to each vertex (the *weight* of this vertex) are all distinct. We look for the minimum of $\max(v)$ among the possible valuations for some particular classes of graphs.

I. INTRODUCTION.

We consider connected graphs $G = (V, E)$ without loops or multiple edges and consider valuation on the edges that allow to distinguish the vertices. There are several variations of this problem we describe below.

Consider a valuation v on the edges of G i.e. a function (coloration) $v : E(G) \to \mathbf{N}^*$. We call $|v(E)|$ the size of the valuation. This coloration is (or not) proper.

It induces a valuation (or coloration) on the vertices of G by

- either the sums $w(x) = \sum_{y \in N(x)} v(xy)$
- or the multisets $S(x) = \{v(xy), y \in N(x)\}$.

Then we add a last restraint

- either two adjacent vertices must have different valuations
- or all the vertices must have different valuations.

This leads to eight possible notions and parameters :

$(p/np, l/g, \Sigma/\Omega)$ that is proper/no proper valuation of the edges in which the vertices are distinguished in a local/global manner with sums/sets and then correspondant parameters $\chi'_{\Omega/\Sigma}(G, p/np, l/g)$: minimun of the maximum value in a $(p/np, l/g, \Sigma/\Omega)$-valuation. Several authors worked on some of these parameters, for instance see [1], [4], [3], [5].

In this article, we focus on $\chi'_\Sigma(G, p, g)$. Call admissible valuation on E any proper valuation $v : E \to \mathbf{N}^*$ distinguishing vertices by sums, that is to say such that for two vertices $x \neq y$, $\sum_{xz \in E} v(xz) \neq \sum_{yz \in E} v(yz)$. There is no such valuation for the graph K_2 so from now on, we assume that $|V(G)| = n \geq 3$.

An admissible valuation of size $|E| = m$ always exists : namely, let $E = \{e_1, e_2, \cdots, e_m\}$ and $f(e_i) = 2^i$. Since, for $x \neq y$, the set of edges incident to x is distinct from the set of edges incident to y, the binary numbers $\sum_{x \in e_j} 2^j$ and $\sum_{y \in e_j} 2^j$ are distinct. Of course, this valuation is very bad in the sense that its maximum is by far too large, even if m values are necessary. For instance, if $G = K_{1,n}$, $n \geq 2$, any two edges are adjacent so every admissible valuation is of size $m = n$, but the values from 1 to n are sufficient for distinguishing vertices by sums.

For v an admissible valuation, and $x \in V$, we call *weight* of x and note $w(x)$ the sum $w(x) = \sum_{x \in e_j} v(e_j)$.

The subset $\{\max(v(E)) \mid v \text{ an admissible valuation on } E\}$ of $\mathbf{N}$ being nonempty has a minimum, which we denote by $\chi'_w(G)$ for simplicity. For instance, $\chi'_w(K_{1,n}) = n$.

Recall that a proper vertex-distinguishing coloration (in short pvdc) of E of size q, is a surjective application $\varphi : E \to \{1, \cdots, q\}$ with the following properties :

- for any two adjacent edges e, e', $\varphi(e) \neq \varphi(e')$
- for any two distinct vertices $x \neq y$ the multisets $\{\varphi(e) \mid x \in e\}$ and $\{\varphi(e) \mid y \in e\}$ are distinct.

Then we have the following :

Theorem 1 *There is an admissible valuation on E of given size, if and only if there is a pvdc of E of the same size.*

PROOF. For a necessary condition, if $v : E \to \mathbf{N}^*$ is an admissible valuation of size q, then any bijection $g : v(E) \to \{1, \cdots, q\}$ induces a pvdc $\varphi = g \circ v$. Conversely, if φ is a pvdc of size q, then $v = 2^{\varphi}$ is an admissible valuation of the same size. ■

The minimum of colors used in a pvdc of E is denoted by $\chi'_s(G)$. We immediately deduce the following :

Corollary 1 *For any graph, we have $\chi'_s(G) \leq \chi'_w(G) \leq 2^{\chi'_s(G)-1}$.*

These bounds are tight. For instance, $\chi'_s(K_{1,n}) = \chi'_w(K_{1,n}) = n$ and $\chi'_w(G) = 4 = 2^{\chi'_s(G)-1}$ for the "extended-3-star" G obtained by identifying the first extremities of three copies of P_3.

II. Other bounds.

We give a lower bound for $\chi'_w(G)$ in the general case, and other bounds cor regular graphs.

Theorem 2 *If G is a graph of order n, with maximum (respectively minimum) degree Δ (respectively δ) then*

$$\chi'_w(G) \geq \left\lceil \frac{n-1}{\Delta} + \frac{\Delta - 1}{2} + \frac{\delta(\delta+1)}{2\Delta} \right\rceil .$$

PROOF. For any admissible valuation v on $E(G)$, there are n distinct weights on the vertices, so the minimum weight w and the maximum weight W satisfy the inequality $n-1 \leq W - w$. On one hand we have in all cases $w \geq 1 + \cdots + \delta = \delta(\delta+1)/2$. On the other hand, if we have $\max(v) = \chi'_w(G)$, then $W \leq (\chi'_w(G) - \Delta + 1) + \cdots + \chi'_w(G) =$

$\Delta(2\chi'_w(G)-\Delta+1)/2$ thus $n-1 \le W-w \le \Delta(\chi'_w(G)-(\Delta-1)/2-\delta(\delta+1)/(2\Delta))$ implying inequality of the theorem. ■

This bound is tight : for instance, we shall show that $\chi'_w(K_{p,p-1}) = p+1$ if $3 \le p \le 8$.

Let G be a d-regular graph, $d \ge 2$, and q any integer≥ 1. A valuation v on E is admissible if and only if the valuation $v+q$ is admissible, since all the weights are increased by dq. Therefore we have :

Proposition 1 *If G is regular and v is an admissible valuation on E with* $\max(v) = \chi'_w(G)$*, then* $\min(v) = 1$.

The following result is almost as obvious :

Proposition 2 *If G is a d-regular graph $\chi'_w(G \square K_2) \le 2\chi'_w(G) - d + 2$.*

PROOF. Let v be an admissible valuation on the edges of G with maximum $\chi'_w(G)$. On one copy of G put $v+1$ so that the minimum is now 2. Since the difference between the maximum and the minimum weights is at most $d(\chi'_w(G)-d)$, by setting $v+(\chi'_w(G)-d+2)$ on the edges of the second copy og G we obtain distinct weights greater than those of the first copy. Now we give value 1 to the edges of the perfect matching corresponding to factor K_2 of the product and we are done. ■

We may slightly improve, for d-regular graphs, the lower bound $d+(n-1)/d$ given in the first theorem of this section by the following result, which is significant when d divides $n-1$:

Theorem 3 *Let G be a d-regular graph of order n. Then we have :*

$$\chi'_w(G) \ge \left\lceil d + \frac{n-1}{d} + \frac{2\epsilon}{nd} \right\rceil$$

with $\epsilon = 1$ if the number $n(2D+n-1)/2$, where $D = d(d+1)/2$, is odd, and $\epsilon = 0$ otherwise.

PROOF. Let v an admissible valuation on E with $\max(v) = \chi'_w(G) = p$, and size $|v(E)| = q$, say $v(E) = \{v_1, \ldots, v_q\}$. For $1 \le i \le q$, let k_i be the number of edges such that $v(e) = v_i$, so we have $\sum_{i=1}^{q} k_i = |E| = nd/2$. The n weights $w(x)$, $x \in V$ are distinct numbers at least equal to $1+\cdots+d = D$. So the total sum of weights is at least $D+\cdots+(D+n-1) = n(2D+n-1)/2$. In this sum, the value v_i appears $2k_i$ times, therefore we obtain, since this sum is even (ϵ being as in the statement of the Theorem) :

$$2\sum_{i=1}^{q} k_i v_i \ge \epsilon + n(2D+n-1)/2.$$

Now, since G is regular, $v' = p+1-v$ is another admissible valuation on E and we have also :

$$2\sum_{i=1}^{q} k_i v'_i \geq \epsilon + n(2D+n-1)/2.$$

Adding these two inequalities, we obtain : $2(p+1)nd/2 \geq 2\epsilon + n(2D+n-1)$ which gives the result. ■

This bound is tight : for instance if G is the well-known Petersen graph, one can easily find an admissible valuation v on its edges with $\max(v) = 7$.

III. Results following constructions for χ'_s.

The construction given in [2] for a proper vertex-distinguishing coloring of the edges of K_n of size $\chi'_s(K_n)$ altogether gives an admissible valuation :

Theorem 4 *We have :*

$$\chi'_w(K_n) = \chi'_s(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$$

PROOF. Recall the construction of [2]. For $k \geq 2$ arrange the vertices of K_{2k} in the form of a regular $(2k-1)$-gon $x_1, \ldots, x_{2k-1}$ with one vertex x_{2k} in the center. The radial edge $(x_{2k}x_i)$ together with the edges perpendicular to it is a perfect matching, to which we give the valuation i. At this step, all the vertices have the same weight.

In order to obtain a K_{2k-1} delete vertex x_1. Since the valuation was proper, the weights of the other vertices decrease by distinct values, which gives the result for n odd.

Now, for $k \geq 3$, delete moreover x_2. It is easy to check that the sums $(v(x_ix_1) + v(x_ix_2))_{3\leq i\leq 2k}$ are all distinct. Therefore we obtain an admissible valuation of K_{2k-2} and the result for n even. ■

IV. Some results on irregular bipartite complete graphs.

We already saw that for $n \geq 2$, $\chi'_s(K_{n,1}) = \chi'_w(K_{n,1}) = n$ with the set of values $\{1, \ldots, n\}$ on the edges. So we concentrate on the graphs $K_{n,p}$ with $n > p \geq 2$.

We shall denote by x_i the vertices of one class (if $n \neq p$, the larger one) and by x'_j those of the other one. Following the process which leads to $\chi'_s(K_{n,p}) = n+1$, we may put, for $1 \leq i \leq n+1$ and $0 \leq j \leq n$ on each edge $x_ix'_{j+i}$ (or $x_ix'_{j+i-n-1}$ if $i+j > n+1$) of a $K_{n+1,n+1}$ the value v_{i+1} in such a way that the set $\{v_i \mid 1 \leq i \leq n+1\}$ equals $\{1, \ldots, n+1\}$, then erase one vertex x_i of the first class, and $n+1-p$ vertices of the other class. Unfortunately, this does not give distinct weights in general. However we have :

Theorem 5 *If p is relatively prime to $n+1$, and $2 \leq p \leq n-3$, then $\chi'_w(K_{n,p}) = \chi'_s(K_{n,p}) = n+1$.*

PROOF. For any integer k, let $\overline{k} = 1 + ((k-1) \ mod \ (n+1))$ that is to say, the unique integer in the range $[1, n+1]$ such that $k - \overline{k}$ is divisible by $n+1$. Let $q = n+1-p$, so q is relatively prime to $n+1$. Put $a = n/2$ if n is even, $a = (n-1)/2$ if $n = 4k+3$ and $a = (n-3)/2$ if $n = 4k+1$. In every case a is relatively prime to $n+1$.

With the above notations, let $v_i = \overline{1 + (i-1)a}$. Since a is relatively prime to $n+1$, the sets $\{v_i \mid 1 \le i \le n+1\}$ and $\{j \mid 1 \le j \le n+1\}$ are equal, so the weights on the edges of the $K_{n+1,n+1}$ are all equal to $W = 1 + \cdots + (n+1) = (n+1)(n+2)/2$. Now we erase the vertex x_1 in the first class, and vertices $x'_i, p+1 \le i \le n+1$ in the other class.

Therefore the weights of the vertices $x'_i, 1 \le i \le p$ decrease respectively by the values v_i, all distinct and all no greater than $n+1$ and the remaining weights w'_i are therefore all distinct. On the other hand, the weights of the x_i decrease since $q \ge 4$ at least by $\overline{1-a} + 1 + (1+a) = n+4$ and the remaining weights w_i are all distinct from the w'_i. For $1 \le i \le n+1$, let $s_i = (1 + (i-1)a) + (1 + ia) + \cdots + (1 + (i+q-2)a)$ and $\tilde{s}_i = v_i + \cdots + v_{\overline{i+q-1}}$, so for $1 \le i < j \le n+1$, $s_j - s_i - (\tilde{s}_j - \tilde{s}_i)$ is divisible by $n+1$, whereas $s_j - s_i = (j-i)qa$ is not, since qa is relatively prime to $n+1$. Thus the $\tilde{s}_i$ are all distinct. Now the weights w_i are n distinct elements in the set $\{W - \tilde{s}_i \mid 1 \le i \le n+1\}$, so we obtain an admissible valuation on the edges of $K_{n,p}$. ■

With other choices of the values v_i, we obtain the following

Theorem 6 *For any $n \ge 4$, $\chi'_w(K_{n,n-2}) = \chi'_s(K_{n,n-2}) = n+1$.*

PROOF. As above, let $W = 1+\cdots+(n+1)$ and $\{v_i \mid 1 \le i \le n+1\} = \{1, \ldots, n+1\}$. For any choice of the values v_i, by erasing vertices x_{n+1}, x'_{n-1}, x'_n and x'_{n+1}, the remaining weights for the other x'_i are all distinct and not smaller than $W - (n+1)$; those of the vertices $x_i, 1 \le i \le n-1$ are the elements of the set $\{W - (v_i = v_{i+1} + v_{i+2}) \mid 1 \le i \le n-1$ and that of the vertex x_n is $W - (v_{n+1} + v_1 + v_2)$. In order to obtain an admissible valuation, it is sufficient that the n sums $v_i + v_{i+1} + v_{i+2}, 1 \le i \le n-1, v_n + v_{n+1} + v_1$ are distinct and all greater than $n+1$. We give in any case a choice satisfying these properties, letting the checking to the reader.

- If $n = 3k-2$, for $1 \le i \le k, v_{3i-2} = i, v_{3i-1} = i+k$ and for $1 \le i \le k-1, v_{3i} = i+2k$.
- If $n = 3k-1$, for $1 \le i \le k, v_{3i-2} = i-1+2k, v_{3i-1} = i$, for $1 \le i \le k-1, v_{3i} = i+k$ and $v_{3k} = 3k$.
- If $n = 3k$, for $1 \le i \le k, v_{3i-2} = i, v_{3i-1} = i+k, v_{3i} = i+2k$ and $v_{n+1} = n+1$.

■

And with some slight modifications, we also obtain the following result :

Theorem 7 *If p satisfies $2 \le p < n - (\sqrt{8n+25} - 5)/2$, then $\chi'_w(K_{n,p}) = \chi'_s(K_{n,p}) = n+1$.*

PROOF. We only need to give a proof when $gcd(n+1,p) = d$ is at least 2 and $p \geq 3$. First begin with valuations $v_i = i$ on the edges of a $K_{n+1,n+1}$, and weight $W = (n+1)(n+2)/2$ for all its vertices. Then erase vertex x_1 of the first class and vertices $x_i, p+1 \leq i \leq n+1$ of the other one. The remaining weights of the second class are $W - i, 1 \leq i \leq p$, all distinct. Those of the first class are the elements of the two sets $\mathcal{W}_1 = \{w_i = i + \cdots + (i+p-1) \mid 2 \leq i \leq n-p+2\}$, and $\mathcal{W}_2 = \{w_j = j+\cdots+(n+1)+1+\cdots+(j+p-n-2) = j + \cdots + (j+p-1) - (j+p-n-2)(n+1) \mid n-p+3 \leq j \leq n+1\}$. The elements of each $\mathcal{W}_i$ are obviously distinct, but it may occur that some w_i in $\mathcal{W}_1$ is equal to some w_j in $\mathcal{W}_2$, which actually is the case. Now add 1 to the valuations $v(x_i x'_p)$ for $2 \leq i \leq p$ and substract p to the valuation $v(x_{n+1}x'_p)$ (the result is 1). By this modification, the weights of the second class become $W - i, 1 \leq i \leq (p-1)$ and $W - (p+1)$. those of $\mathcal{W}_1$ remain unchanged except for the lesser one w_2 replaced by $w_2 - p$ (so these weights remain distinct) and each weight of $\mathcal{W}_2$ is increased by 1, and they remain distinct.

Let define d' by :

- $d' = d$ if d is odd or if $d = 2$ and p divisible by 4.
- $d' = d/2$ otherwise.

Note that we have $d' \geq 2$ except for the case when $d = 2$ and p not divisible by 4, where $d' = 1$. Since each sum $i+\cdots+(i+p-1) = p(2i+p-1)/2$ is divisible by p if p is odd, and by $p/2$ but not by p if p is odd, each weight of $\mathcal{W}_1 \cup \mathcal{W}_2$, before modification is divisible by d' when $d' \leq 2$ (respectively is odd when $d' = 1$) and after modification, this property is preserved for the weights of $\mathcal{W}_1$ but not for the weights of $\mathcal{W}_2$ therefore these modified weights are all distinct. Now, condition $p < n - (\sqrt{8n+25} - 5)/2$ insures that they are distinct from the remaining weights of the other class, since they are at most equal to $(n-p+2) + \cdots + (n+1) = p(2n-p+3)/2$ and the equation $p(2n-p+3) = 2(W-(p+1))$ (that is to say $p^2 - (2n+5)p + (n^2+3n) = 0$) in p has two roots, namely $p_1 = n - (\sqrt{8n+25} - 5)/2$ and $p_2 = n + (\sqrt{8n+25} - 5)/2 > n$. ■

CONJECTURE. For $2 \leq p \leq n-2$ we always have $\chi'_w(K_{n,p}) = n+1$.

On the contrary, whereas for any $n \geq 3$, $\chi'_s(K_{n,n-1}) = n+1$, we have

Theorem 8 *If* $3 \leq n \leq 8$, $\chi'_w(K_{n,n-1}) = n+1$, *but for* $n \geq 9$, $\chi'_w(K_{n,n-1}) \geq n+2$.

PROOF. We can put the values of a valuation on the edges of a $K_{n,p}$ as coefficients of a $(n,p) - matrix V$, namely $v_{(i,j)} = v(x_i x'_j)$. The valuation is admissible if and only if the coefficients in each line or column of V are all distinct and the $n+p$ sums of coefficients of a line or column are all distinct. For $3 \leq n \leq 8$ the following matrices give an admissible valuation with maximum value $n+1$ for the edges of $K_{n,n-1}$:

$$
\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 3 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 3 \\ 5 & 6 & 4 & 2 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ 2 & 6 & 7 & 1 & 3 \\ 3 & 1 & 6 & 2 & 4 \\ 4 & 2 & 1 & 3 & 5 \\ 5 & 7 & 3 & 4 & 6 \\ 6 & 4 & 5 & 7 & 2 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 8 & 3 & 6 & 4 & 2 \\ 2 & 1 & 8 & 7 & 5 & 3 \\ 3 & 5 & 1 & 8 & 6 & 4 \\ 4 & 3 & 2 & 1 & 7 & 5 \\ 5 & 4 & 7 & 2 & 8 & 6 \\ 6 & 2 & 4 & 3 & 1 & 7 \\ 7 & 6 & 5 & 4 & 3 & 8 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 2 & 8 & 9 & 6 & 4 & 3 \\ 2 & 5 & 9 & 8 & 7 & 6 & 4 \\ 3 & 9 & 1 & 7 & 8 & 5 & 2 \\ 4 & 3 & 2 & 1 & 9 & 7 & 5 \\ 5 & 4 & 3 & 2 & 1 & 8 & 6 \\ 6 & 1 & 4 & 3 & 2 & 9 & 7 \\ 7 & 6 & 5 & 4 & 3 & 1 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 9 \end{pmatrix}
$$

Now if there exist an admissible valuation with maximum value $n+1$ on the edges of a $K_{n,n-1}$ the sums of the lines of the associate $(n, n-1)$-matrix are n distinct elements of the set $S = \{w, \ldots, w+2(n-1)\}$ where $w = 1+\cdots+(n-1)$ and those of the columns are $n-1$ distinct elements of the set $G = \{w+n, \ldots, w+2n\}$, the sum σ of the n weights taken in S being equal to the sum Σ of the $n-1$ weights taken in G. Note that we have $G \setminus S = \{w+(2n-1), w+2n\}$ and that $S \cap G$ is of cardinality $n-1$. Let k the number of elements of this set occuring in the sum Σ. If k were 0, the elements occuring in σ would all be in the set $S \setminus G$ and we would have $\sigma < \Sigma$, a contradiction. So $k = 1$ or $k = 2$. For $k = 1$ we have $\Sigma \geq (n-1)w+(2n-1)+(n-2)(3n-3)/2 = (n-1)w+3n^2/2-5n/2+2$ and $\sigma \leq nw + (2n-2) + (n-1)n/2 = (n-1)w + n^2 + n - 2$. Equality $\Sigma = \sigma$ implies $3n^2/2 - 5n/2 + 2 \leq n^2 + n - 2$ so $n \leq 5$.

For $k = 2$ we obtain $\Sigma \geq (n-1)w + 4n - 1 + (n-3)(3n-4)/2 = (n-1)w + 3n^2/2 - 5n/2 + 5$ and $\sigma \leq nw + 4n - 5 + (n-2)(n+1)/2 = (n-1)w + n^2 + 3n - 6$ and equality between the sums implies $3n^2/2 - 5n/2 + 5 \leq n^2 + 3n - 6$ so $n \leq 8$.

The following matrix actually gives an admissible valuation on the edges of $K_{9,8}$ with maximum value 11.

$$
\begin{pmatrix} 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 11 & 9 & 8 & 10 & 2 & 1 & 6 \\ 6 & 10 & 7 & 9 & 1 & 8 & 11 & 4 \\ 7 & 3 & 8 & 6 & 11 & 1 & 2 & 10 \\ 8 & 9 & 10 & 1 & 2 & 7 & 4 & 3 \\ 9 & 2 & 1 & 10 & 4 & 11 & 3 & 5 \\ 10 & 4 & 11 & 2 & 3 & 6 & 5 & 9 \\ 11 & 1 & 4 & 5 & 7 & 3 & 6 & 2 \\ 3 & 8 & 5 & 11 & 9 & 10 & 7 & 1 \end{pmatrix}.
$$

■

CONJECTURE. For $n \geq 9$ we have $\chi'_w(K_{n,n-1}) = n + 2$.

V. The regular bipartite complete graphs.

Theorem 9 *For $n \leq 2$, $\chi'_w(K_{n,n}) = \chi'_s(K_{n,n}) = n+2$.*

PROOF. Let v any valuation on the edges of a $K_{n,n}$ whose values are in the set $E_n = \{1, \cdots n+2\}$. As above, we set the values $v_{i,j} = v(x_i x'_j)$ as coefficients of an $(n,n)-$matrix V. Then v is an admissible valuation if and only if the n lines L_i and the n columns V_i are $2n$ subsets (necessarily distinct) of cardinality n of E with the following properties :

- For any $k \in E$ the sets $\{i \mid k \in L_i\}$ and $\{j \mid k \in C_j\}$ have the same cardinality.
- The $2n$ sums of the coefficients of each line and each column are distinct.

Since the graph is regular, this is equivalent to the fact that the $2n$ complementary subsets $L'_i = L^c_i$ and $C'_j = C^c_j$ are $2n$ subsets of E of cardinality 2 satisfying the same properties.

Thus we can solve the problem in two steps : first give $2n$ subsets of cardinality 2 in E having the required properties, then construct an (n,n)-matrix V such that the sets L'_i (respectively C'_j) are the sets of "missing numbers" in the lines (respectively the columns) of V. This is done in the following for $n \geq 5$ since the following matrices are easily seen as solutions for respectively $n = 2, 3$ and 4 :

$$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 2 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \\ 5 & 6 & 4 & 1 \\ 6 & 4 & 5 & 2 \end{pmatrix}.$$

FIRST STEP. If n is even, take as sets L'_i the $n/2$ sets $\{1, i\}$ with $2 + n/2 \leq i \leq n+1$ together with the $n/2$ sets $\{j, n+2\}$ with $2 \leq j \leq 1+n/2$. As sets C'_j the $n/2$ sets $\{1, i\}$ with $2 \leq i \leq 1+n/2$ together with the $n/2$ subsets $\{j, n+2\}$ with $2+n/2 \leq j \leq n+1$.

If n is odd, for the L'_i take the sets $\{1, i\}$, $(n+3)/2 \leq i \leq n$ and the sets $\{j, n+2\}$, $2 \leq j \leq (n-1)/2$ together with the sets $\{(n+1)/2, (n+3)/2\}$ and $\{n+1, n+2\}$. For the C'_j take the sets $\{1, i\}$, $2 \leq i \leq (n+1)/2$ and the sets $\{j, n+2\}$, $(n+3)/2 \leq j \leq n$ together with the set $\{(n+3)/2, n+1\}$.

In every case, the required properties are easy to check.

SECOND STEP. First define, for k elements $(a_i)_{1 \leq i \leq k}$, a matrix $C(a_1, \ldots, a_k)$ by $\forall (i,j) \in \{1, \ldots, k\}^2$, $c_{(i,j)} = a_{\overline{j-i+1}}$ where $\overline{s}$ is the unique integer in $\{1, \ldots, k\}$ such that $s - \overline{s}$ is divisible by k.

We divide our construction into three cases.

FIRST CASE : n ODD. Let $n = 2k+1$. Put $A_1 = A_2 = C(1, \ldots, k+1)$ and $B_1 = B_2 = C(k+2, \ldots, 2k+2)$. In A_1 replace, for $1 \leq i \leq k+1$, $a_{(i,i)}$ (whose value is 1) by $n+2$, call $\tilde{A}_1$ this new matrix. In B_2 interchange the lines 1 and k, we obtain a new matrix B'_2 ; in this matrix, replace $b'_{(k,k+1)}$ by 1 and $b'_{(k+1,k+1)}$ by $n+2$, name $\tilde{B}_2$ the resultant matrix. Build with these matrices a $(2k+2, 2k+2)$-matrix $V' = \begin{pmatrix} \tilde{A}_1 & B_1 \\ \tilde{B}_2 & A_2 \end{pmatrix}$. Now erasing line $k+1$ and column $2k+1$ of this matrix gives as result a matrix V associated to an admissible valuation for the edges of a $K_{n,n}$.

For instance, if $n = 7$, the result is the following matrix

$$\begin{pmatrix} 9 & 2 & 3 & 4 & 5 & 6 & 8 \\ 4 & 9 & 2 & 3 & 8 & 5 & 7 \\ 3 & 4 & 9 & 2 & 7 & 8 & 6 \\ 7 & 8 & 5 & 6 & 1 & 2 & 4 \\ 8 & 5 & 6 & 7 & 4 & 1 & 3 \\ 5 & 6 & 7 & 1 & 3 & 4 & 2 \\ 6 & 7 & 8 & 9 & 2 & 3 & 1 \end{pmatrix}.$$

Second case : $n = 4k+2$. Put $A = C(1,\dots,2k+1), B = C(2k+2,\dots,4k+2)$. For $2 \le i \le 2k+1$, exchange $a_{(i,2k+3-i)}$ and $b_{(i,2k+3-i)}$ in order to obtain two new matrices A' and B'. Make two copies A'_1, A'_2 of A' and two copies B'_1, B'_2 of B'. In A'_1 replace, for $1 \le i \le 2k+1$, $a'_{(i,i)}$ (whose value is 1) by $n+2$, we obtain a matrix $\tilde{A}$. In B'_1, replace for $1 \le i \le k$ and $k+2 \le i \le 2k+1$, $b'_{(i,2k+2-i)}$ by $n+1$, we obtain the matrix $\tilde{B}_1$. In B'_2 replace $b'_{(1,1)}$ and, for $2 \le i \le 2k+1$, $b'_{(i,2k+3-i)}$ by $n+1$, we obtain a matrix $\tilde{B}_2$. Now the matrix $V = \begin{pmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{B}_2 & A'_2 \end{pmatrix}$ is associated to an admissible valuation on the edges of a $K_{n,n}$.

For instance, for $n = 10$ the result is the following matrix

$$\begin{pmatrix} 12 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 \\ 5 & 12 & 2 & 3 & 9 & 10 & 6 & 7 & 11 & 4 \\ 4 & 5 & 12 & 7 & 3 & 9 & 10 & 6 & 2 & 8 \\ 3 & 4 & 10 & 12 & 2 & 8 & 11 & 5 & 6 & 7 \\ 2 & 8 & 4 & 5 & 12 & 11 & 3 & 9 & 10 & 6 \\ 11 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\ 10 & 6 & 7 & 8 & 11 & 5 & 1 & 2 & 3 & 9 \\ 9 & 10 & 6 & 11 & 8 & 4 & 5 & 1 & 7 & 3 \\ 8 & 9 & 11 & 6 & 7 & 3 & 4 & 10 & 1 & 2 \\ 7 & 11 & 9 & 10 & 6 & 2 & 8 & 4 & 5 & 1 \end{pmatrix}.$$

Third case : $n = 4k$. Put $B = C(2k+1,\dots,4k)$. Construct matrix A by interchanging in $C(1,\dots,2k)$, for $i = k$ and $i = 2k$ the coefficients $c_{(i,1)}$ and $c_{(i,k+1)}$. Exchange between A and B for $2 \le i \le k$, $a_{(i,2k+2-i)}$ with $b_{(i,2k+2-i)}$, for $k+1 \le i \le 2k-1$, $a_{(i,2k+1-i)}$ with $b_{(i,2k+1-i)}$, and at last $a_{(2k,2k-1)}$ with $b_{(2k,2k-1)}$, we obtain two matrices A' and B' of which we make two copies A'_1, A'_2 and B'_1, B'_2. In A'_1, replace for $1 \le i \le 2k, a'_{(i,i)}$ (whose value is 1) by $n+2$ name this matrix $\tilde{A}$. In B'_1 replace respectively, for $1 \le i \le k$, $b'_{(i,2k+1-i)}$ and for $k+1 \le i \le 2k-1$, $b'_{(i,2k-i)}$ by $n+1$, we obtain $\tilde{B}_1$. In B'_2 replace respectively for $2 \le i \le k$, $b'_{(i,2k+2-i)}$, for $k+1 \le i \le 2k-1$, $b'_{(i,2k+1-i)}$ and $b'_{(2k,2k-1)}$ by $n+1$ in order to obtain $\tilde{B}_2$. Now matrix $V = \begin{pmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{B}_2 & A'_2 \end{pmatrix}$ gives a solution for $K_{n,n}$.

For instance, if $n = 8$ we obtain :

$$V = \begin{pmatrix} 10 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\ 2 & 10 & 4 & 7 & 8 & 5 & 9 & 3 \\ 3 & 8 & 10 & 2 & 9 & 4 & 5 & 6 \\ 4 & 3 & 8 & 10 & 6 & 7 & 2 & 5 \\ 9 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 8 & 5 & 6 & 9 & 2 & 1 & 4 & 7 \\ 7 & 9 & 5 & 6 & 3 & 8 & 1 & 2 \\ 6 & 7 & 9 & 5 & 4 & 3 & 8 & 1 \end{pmatrix}.$$

■

VI. Exact values for the cycles.

It is not always a valuation of minimum size $\chi'_s(G)$ which gives the minimum $\chi'_w(G)$. For instance, an admissible valuation of $G = C_{36}$ of size $\chi'_s(G) = 9$ induces as weights all the combinations $v_i + v_j, i \neq j$, and these sums must be all distinct, implying $\max(v) > 20$, whereas we show in this section that $\chi'_w(G) = 20$.

Theorem 10

- $\chi'_w(C_n) = (n+4)/2$, *if n is even*
- $\chi'_w(C_n) = (n+5)/2 = 2k+3$ *if $n = 4k+1$*
- $\chi'_w(C_n) = (n+3)/2 = 2k+3$ *if $n = 4k+3$*

PROOF. The cycle C_n of order n is a 2-regular graph, giving $D = 3$ and $2D + n - 1 = n + 5$, the number $n(n+5)/2$ is odd if and only if $n = 4k+1$ or $4k+2$. Theorem deux gives $\chi'_w(G) \geq 2 + (n-1)/2 + \epsilon/n$, whence $\chi'_w(G) \geq (n+4)/2$ if n is even, $\chi'_w(G) \geq (n+3)/2$ if $n = 4k+3$ and $\chi'_w(G) \geq (n+5)/2$ if $n = 4k+1$ since in this case $\epsilon = 1$.

In every case we exhibit an admissible valuation with equality holding for the maximum. Let $e_1, \ldots, e_n$ be the clockwise sequence of the edges. We give the sequence $(v) = (v(e_1), \cdots, v(e_n))$.

- For $n = 3$, $(v) = (1, 2, 3)$
- For $n = 4$, $(v) = (1, 3, 4, 2)$
- For $n = 5$, $(v) = (1, 2, 3, 4, 5)$
- For $n = 6$, $(v) = (1, 3, 2, 4, 5, 2)$
- For $n = 4k+3, k \geq 1$, $(v) = (2k+3, 1, 2, (2k+3, 2i+1, 1, 2i+2)_{1 \leq i \leq k})$
- For $n = 4k, k \geq 2$, $(v) = ((2k+1, 2i, 1, 2i+1)_{1 \leq i \leq k-1}, 2k+1, 2k+2, 2k, 1)$
- For $n = 4k+1, k \geq 2$, $(v) = ((2k+2, 2i, 1, 2i+1)_{1 \leq i \leq k}, 2k+3)$
- For $n = 4k+2, k \geq 2$, $(v) = (2k+2, 1, 2, (2k+2, 2i+1, 1, 2i+2)_{1 \leq i \leq k-1}, 2k+2, 2k+1, 2k+3)$

For instance, the clockwise sequence of the values on the edges of C_{18} is $(10, 1, 2, 10, 3, 1, 4, 10, 5, 1, 6, 10, 7, 1, 8, 10, 9, 11)$ which gives an admissible valuation with maximum $11 = (n + 4)/2$. ■

VII. Exact values for the paths.

The results are very similar to those for the cycles. Call as usual, $P_n, n \geq 3$ the path on n vertices $V = \{x_1, \dots, x_n\}$ with edges $e_i = (x_i, x_{i+1})$, for $1 \leq i \leq n - 1$.

Lemma 1

- $\chi'_w(P_n) \geq (n + 2)/2$ *if n is even*
- $\chi'_w(P_n) \geq (n + 3)/2$ *if $n = 4k + 1$*
- $\chi'_w(P_n) \geq (n + 1)/2$ *if $n = 4k + 3, k \geq 2$*
- $\chi'_w(P_7) \geq 5$

<u>Proof.</u> Let v be an admissible valuation on E with $\max(v) = \chi'_w(P_n) = p$. Close the path into a cycle by adding an edge $e_0 = (x_n, x_1)$ and extend the valuation into a valuation $\tilde{v}$ on $E \cup \{e_0\}$ by setting $\tilde{v}_0 = \tilde{v}(e_0) = 0$. This extension is *not* an admissible valuation, since we admit the value zero, but $\tilde{v} + 1$ is. Therefore $\chi'_w(P_n) + 1 \geq \chi'_w(C_n)$ which gives according to Theorem trois, the required inequality in the general case.

As regards the special case $n = 7$, take an admissible valuation v with $\max(v) = \chi'_w(P_7) = p$. The seven weights are distinct numbers, all at least 1. If the value v_i of v is attributed to k_i edges, we have $2\sum k_i v_i \geq n(n + 1)/2 = 28$ and $\sum k_i = 6$. With the above notation, since $\tilde{v}_0 = 0$, we have also $2\sum k_i \tilde{v}_i \geq 28$, now $\sum k_i = 7$. But $v' = p+1-\tilde{v}$ is an admissible valuation of C_7, thus $2\sum k_i v'_i \geq n(n+5)/2 = 42$. Adding these two inequalities gives $2 \times 7 \times (p+1) \geq 70$ thus $p \geq 4$. We can have equality only if the second inequality is an equality, which implies that the seven weights (on the vertices of C_7) are in fact the numbers from 1 to 7. But the only possible decompositions of six among them for $p = 4$ are : $1 = 0+1, 2 = 0+2, 3 = 1+2, 4 = 1+3, 6 = 2+4, 7 = 3+4$ whereas for 5 there are two possible decompositons. If we choose $5 = 1+4$ (respectively $5 = 2 + 3$), then the value 2 (resp. 1) would appear three times, a contradiction since there must be an even occurence of each value in the set of weights. ■

<u>**Theorem**</u> **11** *Inequalities of the previous lemma are equalities.*

<u>Proof.</u> As for the cycles we exhibit sequences (v) giving the values taken by an admissible valuation on the sequence $(e_1, \dots, e_{n-1})$ with maximum value respectively equal to $(n + 2)/2, (n + 3)/2, 5$ when n is respectively even, equal to $4k + 3$ or to 7.

- For $n = 3$, $(v) = (1, 2)$
- For $n = 4$, $(v) = (1, 3, 2)$
- For $n = 5$, $(v) = (1, 3, 4, 2)$

- For $n = 6,\ (v) = (1, 2, 4, 3, 2)$
- For $n = 7,\ (v) = (1, 2, 3, 1, 5, 2)$
- For $n = 4k, k \geq 2,\ (v) = (2, (2k+1, 2i-1, 1, 2i)_{2 \leq i \leq k}, 2k+1, 1)$
- For $n = 4k+1, k \geq 2,\ (v) = (2, (2k+2, 2i-1, 1, 2i)_{2 \leq i \leq k}, 2k+2, 2k+1, 1)$
- For $n = 4k+2, k \geq 2,\ (v) = (2, 1, 3, (2k+2, 2i, 1, 2i+1)_{2 \leq i \leq k}, 2k+2, 1)$.

It remains the case $n = 4k+3, k \geq 2$. With 1 and 2, the sums $1+i$ for $2 \leq i \leq 2k+2$ and $(2k+2)+i, 2 \leq i \leq 2k+1$ gives all the numbers from 1 to $4k+3$. If we replace $1+(k+2)$ by $2+(k+1)$ and $1+(2k+2)$ by $(k+1)+(k+2)$ we obtain a set of weights in which each value appears an even number of times. Consider the graph G with set of vertices $\{1, \dots, 2k+2\}$ and edges $(1, i)_{2 \leq i \leq k+1 \cup k+3 \leq i \leq 2k+1}, (2, k+1), (k+1, k+2), (2k+2, i)_{2 \leq i \leq 2k+1}$. This is a simple connected graph in which all vertices but 1 and 2 have even degree, the degrees of 1 and 2 being respectively $2k-1$ and 3. This graph has therefore an euclidean path joining 1 to 2. The sequence of numbers in this path gives a sequence of values for an admissible valuation of P_{4k+3}. For instance, if $n = 11$ we may find the sequence $(1, 5, 6, 4, 3, 6, 2, 3, 1, 2)$. ■

References

[1] A.C. Burris,Vertex-distinguishing edge-colorings, Phd Thesis, Memphis state University, (1993).

[2] A.C. Burris and R.H. Schelp, Vertex-distinguishing proper edge-colorings, Journal of Graph Theory, 26(2)(1997), 73-82.

[3] G. Chartrand et al., Irregular networks, Congressus Numerantium, 64(1988), 187-192.

[4] J. Lehel, Facts and quests on degree irregular assignments, Graph Theory, Combinatorics, and Applications, (1991), 765-782.

[5] O. Togni, Force des graphes - Indice optique des réseaux, Phd Thesis, Université de Bordeaux-I, (1998).