RAPPORT DE RECHERCHE

L R I

ON THE COMPLEXITY OF A SELF-STABILIZING SPANNING TREE ALGORITHM FOR LARGE SCALE SYSTEMS

CLEMENT J / HERAULT T / MESSIKA S / PERES O

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

02/2008

Rapport de Recherche N° 1488

CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

On the Complexity of a Self-stabilizing Spanning Tree Algorithm for Large Scale Systems

Julien Clement*, Thomas Herault, Stephane Messika and Olivier Peres
{julien.clement,thomas.herault,stephane.messika,olivier.peres}@lri.fr

Univ Paris Sud; LRI; CNRS; INRIA; Orsay F-91405
Bat 490, Universite Paris-Sud, 91405 Orsay Cedex, France
Phone: +33 1 69 15 69 06, Fax: +33 1 69 15 65 86

Abstract. Many large scale systems, like grids and structured peer to peer systems, operate on a constrained topology. Since underlying networks do not expose the real topology to the applications, an algorithm should build and maintain a virtual topology for the application. This algorithm has to bootstrap the system and react to the arrival and departures of processes.
In a previous article, we introduced a computing model designed for scalability in which we gave a self-stabilizing algorithm that builds a spanning tree. At that time, we provided a proof of stabilisation and performance measurements of a prototypal implementation. In this work, we present a probabilistic method to evaluate the theoretical performances of algorithms in this model, and provide a probabilistic analysis of the convergence time of the algorithm.

Keywords: distributed systems, self-stabilisation, overlay network, time complexity, probabilistic analysis.

1 Introduction

Large scale systems are made of a large number of machines linked by a network. In the case of grids or peer to peer systems, the network is generally based on interconnected local area networks or the infrastructure of Internet service providers. Nodes communicate with each other using a network protocol, like IP, that routes the packets hop by hop.

As a result, it is necessary for large scale systems to operate without knowing the underlying topology. Modern peer to peer systems rely on peer sampling services [9] that allow each node to know a small number of other nodes, ensuring that the resulting topology is connected. They can then either operate directly using this service or build a more constrained topology, like a Chord ring [11].

This model is different from the usual model used to write distributed algorithms. Classically, each process knows the whole set of its neighbors, updated whenever a node arrives or leaves the system. It can be used directly to write distributed algorithms that can operate on large scale systems [8]. The previous works on this subject showed an

* Supported by Région Île de France

algorithm that builds a spanning tree using only a constant number of process identifiers, with a formal proof of self-stabilization. Its performance was evaluated on a prototype implementation.

The experiments showed that the convergence time of the algorithm varies significantly according to the properties of the oracle. We therefore present our main contribution: the expected convergence time of the algorithm using Markov chains methods [6, 12]. We distinguish between two oracles with different behaviors: the first one relies on a uniform distribution of the identifiers and the second one follows a power law.

The rest of the paper is organized as follows. In section 2, we discuss related works. We present the computational model in section 3, the algorithm in section 4, an adapted model for the probabilistic study in section 5, and the probabilistic analysis in section 6. We conclude in section 7.

2 Related Works and Motivation

The performance of overlay networks in large scale systems are typically evaluated probabilistically. For example, the authors of Chord [11] show that the expected number of steps for lookups on their skip ring is $O(\log N)$, where N is the total number of processes in the systems. Similar studies were conducted for other overlay networks, like Pastry [10] or Tapestry [13]. The underlying assumption is that since the system is very large, the identifiers are well spread by a hash function and the running time is long, the whole system globally behaves according to the probabilistic model.

We showed [8] that it is possible to build an overlay network – a spanning tree, in this case – in a large scale system, while keeping in mind the scalability issues, and provide a formal proof of self-stabilization under a nondeterministic scheduler, as opposed to a probabilistic proof of convergence. For this purpose, we introduced a computing model designed for scalability. By avoiding to provide each node with a list of its neighbors, it allows to write algorithms that focus on local operations and make use of an oracle when necessary, to ensure a global connectivity.

Since it is proven that the algorithm eventually stabilizes in any execution, the remaining question is its theoretical performance. We therefore define a probabilistic model and compute the expected convergence time of the algorithm.

The model distributes the knowledge of the global topology in the whole system. In peer to peer systems, this is a common assumption that is typically implemented using a peer sampling service. In this context, each process has a function getPeer() that returns the identifier of another process. Using this function, the nodes get to know each other.

Until recently, it was typically assumed that peer sampling services draw process identifiers following a uniform distribution, i.e. each process has the same probability of being drawn as any other. Recent works on this subject [9] demonstrated that in reality, the main implementations currently in use do not exhibit this behavior. Rather, the knowledge graph looks like a small world system, and the distribution of the answers of the function getPeer() looks like a power law. This led us to consider both distributions in the rest of the article.

3 Model

The algorithm operates in an asynchronous system where each process has local variables, a set of guarded rules and a lossless FIFO link towards all the other processes. Processes do not know their neighbors a priori, so p can send a message to q if and only if the identifier of q is in a local variable of p. The capacity of each channel is bounded by an unknown constant, and like Afek and Bremler [1], we assume that this issue can be addressed by buffering a constant number of messages if necessary. The scheduler is nondeterministic and fair in the sense that any rule whose guard evaluates to true in an infinite suffix is eventually drawn.

Each process has a unique identifier in a domain $\mathcal{I}$ that is divided into two subdomains $\Pi \subseteq \mathcal{I}$ contains the identifiers of *correct* process, i.e. those that do not crash, and the other identifiers belong to crashed processes or to no process at all. However, a given process cannot know whether some $i \in \mathcal{I}$ is in Π or not. A total order $<$ on $\mathcal{I}$ is available, as on most large scale systems: for example, IP addresses are totally ordered.

To provide the processes with the ability to connect the whole system, each of them has a local device called an *oracle*. Each process can query its oracle as part of an action by calling the function get_peer(). The answer is an identifier in $\mathcal{I}$. The global condition on the set of all oracles in the system is that in an execution, if $\mathcal{S}$ is the set of all processes that query their oracles an infinite number of times, then all the processes in $\mathcal{S}$ eventually obtain all the identifiers of $\mathcal{S}$. This device is an adapted version of the concept of peer sampling service, tailored for the needs of the algorithm.

To make it possible to actually use this system, the processes need to be able to eventually decide whether a process is live or not. Since Fischer, Lynch and Patterson's theorem [5] states that the consensus problem is unsolvable in an asynchronous system where a crash is possible, we use failure detectors. Each process has one, following the definition given by Chandra and Toueg [2]. This local device provided a predicate suspect: $\Pi \mapsto \mathrm{boolean}$. In this papers, all the detectors are in class $\diamond\mathcal{P}$, which means that after a finite but unknown time, for all processes, $\forall p$, suspect(p) $\implies$ process p is crashed.

The algorithm makes use of self-stabilization [3]. This technique, introduced by Dijkstra [3] and since then adapted to many different contexts [4], allows a system to recover from any sequence of transient failures. To account for this, the whole system, comprising the processes and the channels, is initialized arbitrarily. Then, the runs are failure-free.

4 The Algorithm

This is a simplified version of the algorithm. It is sufficient to understand how the tree is built. The complete algorithm is given in the previous paper [8].

The goal is to build a spanning tree whose degree is bounded by a constant δ. To make sure that there can be no cycle, the algorithm enforces a global invariant: the identifier of each process is higher than those its children and lower than that of its parent.

Each process has local variables, an identifier ($myself$) and a set of guarded rules.

Variables
parent: $\Pi \cup \{\perp\}$ ($\perp$ means no value)
children: set of Π
Guarded rules
the spontaneous rule, in practice, is regularly triggered by the scheduler and mostly takes care of cleanup and connectivity.
true $\rightarrow$
delete any incorrect child: greater than myself, cardinal of the child set bigger than δ, suspect child ;
if father = $\perp$ (i.e. this process is root), pick a process given by the oracle and send it a message to try to merge the subtrees.
reception of a merge message
merge message available $\rightarrow$
if this process is root and lower than the sender, it sends back a merge agreement message and take the sender as its new parent.
reception of a merge agreement message
merge agreement message available $\rightarrow$
If possible, take the sender as a child. If there are already δ children, first delete a child lower than the sender. If all δ children are higher than the sender, pass the merge agreement down to one of the children.
link maintenance
true $\rightarrow$
Send a keepalive message to parent and children
keepalive message available $\rightarrow$
if the sender neither parent nor a child, take it as my new parent or child, if possible ; else send back a link rupture message. On reception of such a message, delete the sender from the children list or the parent field.

Fig. 1. The Algorithm

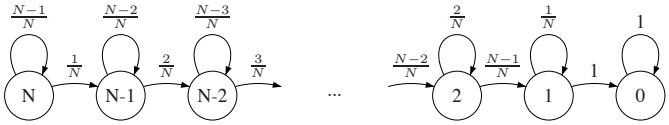


Fig. 2. Model for $\delta = 1$

5 Model for the Probabilistic Study

The above model is well suited for a proof of convergence in all possible cases. However, for a probabilistic study of the convergence time, we switch to a simpler model.

First, we assume the existence of a mapping from Π to $\mathbb{N}^*$ that assigns consecutive integers to the processes. The process that has the lowest identifier in Π gets number 1, the immediately higher process gets 2, and so on. The highest process is numbered N. This mapping, which is normally not available in a large scale system, is only used for the presentation of the probabilistic analysis and not by the algorithm. We also assume that only the identifiers of live processes are drawn; this does not change fundamentally the results but makes the following explanations easier to understand.

Then, we switch to a new scheduler that uses a coarse-grained notion of rounds. Rounds are the time unit in the rest of this paper.

Definition 1 (Round). *A round is a minimal sequence of consecutive events during which the following happens:*

1. *each process executes its spontaneous guarded rule at least once ;*

2. *all messages sent as part of (1) are received and the corresponding guarded rules are executed ;*
3. *all message sent as part of (2) are received and the corresponding guarded rules are executed. This applies recursively to the duplicated merge agreement messages.*

The existing proof of self-stabilisation allows us to switch to this definition since it implies that the number of actions in a round is necessarily finite.

We define the start of round 1 as the first action in an execution and, recursively, the start of round $i > 1$ as the first action following the last action of round $i - 1$.

6 An Upper Bound on the Convergence Time

We divide the discussion according to the value of δ, the bound on the degree of the spanning tree we are building. Nevertheless, we follow the same guiding thread for the different cases. The idea, as proposed by Gouda [7], is to define a metric over the set of configurations. Indeed, usually, in order to prove that a randomized self-stabilizing algorithm $\mathcal{A}$ reaches a set $\mathcal{L}$ of legal configurations in finite time with probability 1 starting from an arbitrary initial configuration, one exhibits a potential function that measures the distance of any configuration to $\mathcal{L}$, such that this potential function decreases with non-zero probability at each step of $\mathcal{A}$. In each subsection, we define an appropriate metric (potential function) according to δ, the bound on the degree. Once the metric reaches its minimum, the system has converged.

In two separate subsections, we consider first an oracle that uses a uniform distribution, then an oracle that follows a power law.

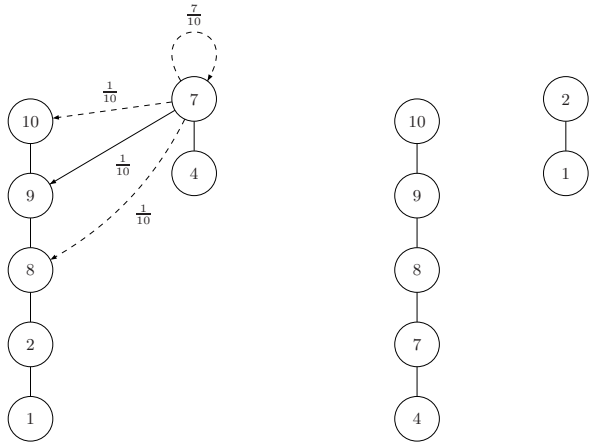
6.1 Uniform distribution

In this section, we assume that the oracle may choose each identifier with probability $1/N$.

Case $\delta = 1$ Since $\delta = 1$, each node may have only one son. Therefore, the main idea is that only one tree is a solution to the problem. It is in fact a chain of processes, ranked from the highest identifier to the lowest. We define that a process is at its *right place* iff 1) the root of its tree is Max and 2) its parent is the smallest higher process and its child is either $\perp$ or the highest lower process. Once a process is at his right place, it will never change position. We define the following metric: $\mu =$(Number of processes that are not at their right place).

Let us study the worst scenario: during each round, only one process joins the final tree, so that the number of processes at their wrong place decreases by one at each round. We model the problem as shown in figure 2, where a *state* represents the number of processes that are not at their right place.

During each round, the process with the highest identifier that is not in the final tree queries the oracle and has a non-zero probability to join the final tree. For example, let us assume there are k processes at the wrong place. Therefore we know that there are already $N - k$ processes in the tree. When P_0, the process with the highest identifier



(a) $\delta = 1, N = 10$, Number of processes at their right place$= 3$. Insertion of a subtree.

(b) $\delta = 1, N = 10$, Number of processes at their right place$= 4$. After the insertion: the metric has decreased.

Fig. 3. Two Examples

among the $N-k-1$ processes not in the final tree, executes the algorithm, if the oracle answers one of the $N-k$ process with a higher identifier, then P_0 joins the final tree. Otherwise, nothing happens. Therefore, it is enough that the oracle chooses one of the $N-k$ process with a higher identifier for the metric to decrease. Such an event has probability $(N-k)/N$ to occur.

An example is shown on figure 3(a). We see that μ = 10-Number of processes at their right place = 10-3 = 7. Then process 7 sends a message to process 9. This means that after having queried its oracle, the answer was 9. According to the algorithm, the maximal degree of the tree being reached, process 9 looks at the identifier of his son. Its child has an identifier higher than 7. Therefore the new process does not replace its son; instead, the information is passed down to process 8. This one has a child with a lower identifier than 7, so a replacement is done. The result is shown on figure 3(b), where $\mu = 10 - 4 = 6$.

To compute the expected time for μ to reach its minimum, we introduce the following stochastic process: $\forall k \in \mathbb{N}, X_k =$ Number of processes not at their right place at time k.

Proposition 1. *$(X_n)_{n\in\mathbb{N}}$ is a Markov Chain.*

Proof sketch: It is easy to see that the number of processes correctly placed on the tree at time $t+1$ only depends on how many of them were at their right place at time t.

Remark 1. This ensures that the algorithm converges with probability 1.

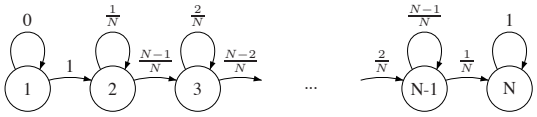


Fig. 4. Model for $\delta = N$

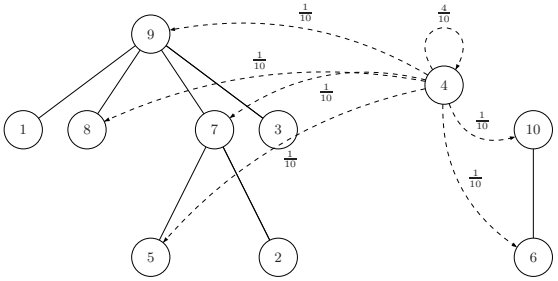


Fig. 5. Example for $\delta = N$

Definition 2. *The expected time for the chain to reach state k, knowing its initial position was l, is:* $\forall k, l \in [0, \dots, N], T_k^l = \mathbb{E}\left[x \backslash X_x = k \text{ knowing } X_0 = l\right]$.

Using these definitions, we obtain the following induction formula: $T_0^{k+1} = 1 + \frac{N-k}{N} T_0^k + \frac{k}{N} T_0^{k+1}$ which leads to $(T_0^{k+1} - T_0^k) = \frac{N}{N-k}$. After summing from 0 to $N-1$ (we notice that $T_0^0 = 0$), we have: $T_0^N = N \sum_{i=0}^{N-1} \frac{1}{N-i} \sim_{N \mapsto +\infty} N \log(N)$.

Case $\boldsymbol{\delta = N}$ This is the second marginal case. We need to define a different metric: $\mu = (N - \text{Identifier of the smallest root})$. The tree is built as soon as there is only one root, i.e. when the identifier of the smallest root is N, thus when $\mu = N - N = 0$. Again, we study the worst-case scenario, i.e. at each round only the smallest root joins the final tree. This is illustrated in figure 4, where a *state* represents the identifier of the smallest root. Assume k is the smallest root: when k queries the oracle, it is enough that it answers one of the $N - k$ higher identifiers for the metric to decrease. This justifies the probability transitions that appear on the figure.

We introduce the following stochastic process to compute the expected time for μ to reach 0: $\forall k \in \mathbb{N}, X_k =$ the identifier of the smallest root at time k.

Proposition 2. $(X_n)_{n \in \mathbb{N}}$ *is a Markov Chain.*

The proof is similar to that made in the case $\delta = 1$.

The computation of the convergence time is similar to what we did in the case $\delta = 1$. Using the previous definition: $\forall k \in [0, \dots, N]\, T_N^k = \mathbb{E}\left[x \backslash X_x = N \text{ knowing } X_0 = k\right]$, we obtain: $T_N^k = 1 + \frac{N-k}{N} T_N^{k+1} + \frac{k}{N} T_N^k$ which leads to $(T_N^k - T_N^{k+1}) = \frac{N}{N-k}$.

Therefore, the expected convergence time is $T_N^0 = N \sum_{i=0}^{N-1} \frac{1}{N-i} \sim_{N \mapsto +\infty} N \log(N)$.

On the example given in figure 5, we see that the process with the smallest identifier has a non-zero probability to join the tree.

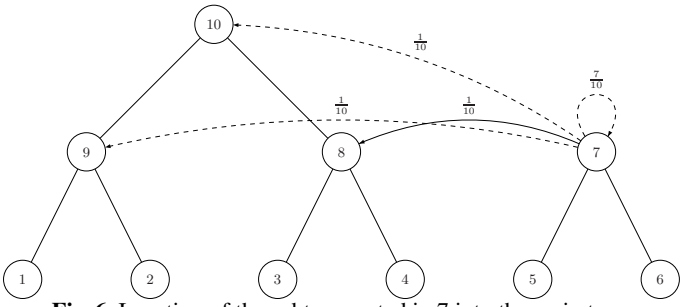


Fig. 6. Insertion of the subtree rooted in 7 into the main tree

Case $2 \leq \delta \leq N-1$ We now turn to the main case, that is $2 \leq \delta \leq N-1$. We define the following metric: $\mu =$ (Number of roots,identifier of the smallest root).

The minimum of this metric is (1,N). When it is reached, there is exactly one root, namely Max, and convergence is achieved. This metric cannot increase since it is not possible in the algorithm to drop two children simultaneously. We now show why it decreases. The idea is that during each round, either the first component decreases, or the second one does. If the number of roots decreases then, obviously, the metric decreases. Otherwise, a root tried to join a tree and took the place of another process. According to the algorithm, a process can only replace a child p with a process q s.t. $q > p$. This indicates that the new root necessarily has a smaller identifier than that of the former root. The second component of the metric decreases.

Figure 6 shows the possible cases for the insertion of a subtree into the main tree.

We can now compute the expected time before this metric reaches its minimum. The worst case goes as follows: the first component stays constant until the second one reaches its minimum, then the number of roots (the first component of the metric) decreases by 1 and the second reaches its "new" maximum, and so on.

Therefore, in order to compute the worst-case convergence time, we need evaluate the time needed for the second component of the metric to reach its minimum, i.e. the time after which no process smaller than the smallest root can become root. In order to study the convergence time, we split the calculations into two phases.

When we studied the case where $\delta = N$, we showed that any process k becomes the smallest root in the system in at most $(N-k)\log(N-k)$ rounds. Thus, since at most N processes will go through this number of rounds, the total convergence time is equivalent to: $N + \sum_{k=0}^{N}(N-k)\log(N-k) \sim_{N\mapsto\infty} N^2 \log(N)$.

Proof sketch: The idea is to use the equivalence between sum and integral:

$\sum_{k=0}^{N}(N-k)\log(N-k) \sim_{N\mapsto\infty} \int_0^N x \log x \mathrm{d}x$.

We obtain: $\int_0^N x\log x = \left[x^2 \log x\right]_0^N - \int_0^N x dx$, which is enough to lead to the result.

6.2 A more realistic distribution

In a state-of-the-art large scale system, the global knowledge of the whole set of process identifiers is distributed among all the processes. The usual device that connects the processes to the rest of the system is a *peer sampling service*. It provides each process with a small set of live process identifiers and ensures that the global knowledge graph is connected.

Recent works in this field [9] has shown that the resulting graph does not follow a uniform distribution, as was previously assumed, but rather a power law. Informally, a realistic oracle based on such a service should give each process p a very high probability of obtaining the identifiers of a small set of processes, which can be seen as "close" to p, and a very low, but non-zero, probability of obtaining any other process.

In this section, the answers of the oracle are distributed as follows: $k \in [\![1, \lfloor \frac{N}{2} \rfloor [\![$ and $\alpha \in]0, 1[$ are given. If process i queries the oracle, it returns process j with probability p_{ij} such that $p_{ij} = \alpha/2k$ if $|i - j| \leq k \mod(N), (1 - \alpha)/(N - 2k)$ otherwise.

The main idea is to give α a large value (close to 1) to introduce a bias in favor of the nearby processes.

In order to complete the complexity study, we must now compute the convergence time of the protocol assuming that the oracle follows this distribution. To simplify the calculations, we assume without loss of generality that k is 1.

Case $\delta = 1$ We use the same metric as for the uniform distribution: $\mu =$(Number of processes that are not in their right place). This measure converges toward 0. We now compute the convergence time.

We define $\beta_k = \frac{\alpha}{2} + \frac{N-k-1}{N-2}(1 - \alpha)$.

Using the same technique as in the previous calculations, we obtain that $T_0^{k+1} = 1 + \beta_k T_0^k + (1 - \beta_k)T_0^{k+1}$. So: $T_0^{k+1} - T_0^k = \frac{1}{\beta_k}$. By summation over k, we obtain $T_0^N \sim N ln \frac{2+\alpha}{2-\alpha}$.

We observe that convergence is achieved faster. Moreover, once a process asks the oracle for a new value, it has a greater chance to obtain exactly the right one (i.e. the immediately higher identifier), which decreases the number of delays in the algorithm.

Case $\delta = N$ Again, we reuse the same metric: $\mu =$(N-Identifier of the smallest root) in order to compute the convergence time.

The scenario is exactly the same as in the previous section. The convergence time towards N for μ is: $T_N^0 \sim N ln \frac{2+\alpha}{2-\alpha}$, which is once again better than the convergence time in the uniform distribution case.

Case $2 \leq \delta \leq N - 1$ This case is more complex, but we take again the same metric: $\mu =$(Number of roots, Identifier of the smallest root).

As usual, the second component of the metric first decreases to its minimum, then the first one decreases.

In the same way, the convergence time is equivalent to $ln \frac{2+\alpha}{2-\alpha} \sum_{k=0}^{k=N} (N - k)$ which is equivalent to $N^2 \ln \frac{2+\alpha}{2-\alpha}$.

Here again, the convergence time is slightly better. This improvement in the convergence time is an encouraging result from a practical point of view. Indeed, it was widely believed that a uniform distribution was good in this context while in fact, the power law that can be observed on real systems yields better results.

7 Conclusion

We studied the theoretical convergence time of a self-stabilizing spanning tree algorithm for large scale systems from a probabilistic point of view. We first showed how it behaves in under the standard hypothesis that a process could learn the identity of its peers following a uniform distribution. Then, we switched to the more realistic hypothesis that the underlying topology is a small world network in which peers are discovered following a power law, where convergence is achieved faster.

References

1. Y. Afek and A. Bremler. Self-stabilizing unidirectional network algorithms by power-supply. In *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA97)*, pages 111–120, 1997.
2. T. D. Chandra and S. Toueg. Unreliable failure detectors for reliable distributed systems. *Journal of the ACM*, 43, March 1996.
3. E. Dijkstra. Self stabilizing systems in spite of distributed control. *Communications of the Association of the Computing Machinery*, 17(11):643–644, 1974.
4. S. Dolev. *Self-Stabilization*. MIT Press, 2000.
5. M. J. Fischer, N. A. Lynch, and M. S. Paterson. Impossibility of distributed consensus with one faulty process. *J. ACM*, 32(2):374–382, 1985.
6. L. Fribourg, S. Messika, and C. Picaronny. Coupling and self-stabilization. *Distributed Computing*, 18(3):221–232, Feb. 2006.
7. M. Gouda. The triumph and tribulation of system stabilization. In *WDAG95 Distributed Algorithms 9th International Workshop Proceedings, Springer LNCS:972*, pages 1–18, 1995.
8. T. Herault, P. Lemarinier, O. Peres, L. Pilard, and J. Beauquier. A model for large scale self-stabilization. In *21st IEEE International Parallel & Distributed Processing Symposium (IPDPS)*, 2007.
9. M. Jelasity, R. Guerraoui, A.-M. Kermarrec, and M. van Steen. The peer sampling service: experimental evaluation of unstructured gossip-based implementations. In *Middleware '04: Proceedings of the 5th ACM/IFIP/USENIX international conference on Middleware*, pages 79–98, New York, NY, USA, 2004. Springer-Verlag New York, Inc.
10. A. Rowstron and P. Druschel. Pastry: Scalable, distributed object location and routing for large-scale peer-to-peer systems. In *IFIP/ACM International Conference on Distributed Systems Platforms (Middleware)*, pages 329–350, Nov. 2001.
11. I. Stoica, R. Morris, D. Karger, M. F. Kaashoek, and H. Balakrishnan. Chord: A scalable peer-to-peer lookup service for internet applications. In *SIGCOMM '01: Proceedings of the 2001 conference on Applications, technologies, architectures, and protocols for computer communications*, pages 149–160, New York, NY, USA, 2001. ACM Press.
12. G. Winkler. P. brémaud: Markov chains: Gibbs fields, monte carlo simulation, and queues. springer verlag, 1999 - reviewed for metrika.
13. B. Y. Zhao, L. Huang, S. C. Rhea, J. Stribling, A. D. Joseph, and J. D. Kubiatowicz. Tapestry: A resilient global-scale overlay for rapid service deployment. *IEEE J-SAC*, 22(1):41–53, January 2004.