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Color degree and alternating cycles in edge-colored graphs *

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Abstract

Given a graph G and an edge coloring C of G, an alternating cycle of G is such a cycle of G in which any adjacent edges have distinct colors. Let $d^c(v)$, named the color degree of a vertex v, be defined as the maximum number of edges incident with v, that have distinct colors. In this paper, some color degree conditions for the existence of alternating cycles of length 3 or 4 are obtained. We also give a bound on the length of a maximum alternating cycle under conditions of color degrees.

Keywords: alternating cycle, color neighborhood, color degree

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1 Introduction and notation

We use [4] for terminology and notations not defined here. Let $G = (V, E)$ be a graph. An *edge-coloring* of G is a function $C : E \rightarrow N$(N is the set of nonnegative integers). If G is assigned such a coloring C, then we say that G is an *edge-colored graph*, or simply *colored graph*. Denote by (G, C) the graph G together with the coloring C and by $C(e)$ the *color* of the edge $e \in E$. For a subgraph H of G, let $C(H) = \{C(e) : e \in E(H)\}$ and $c(H) = |C(H)|$. For a color $i \in C(H)$, let $i_H = |\{e : C(e) = i \text{ and } e \in E(H)\}|$ and say that *color i appears i_H times in H*. For an edge colored graph G, if $c(G) = c$, we call it a *c-edge colored* graph.

For a vertex $v \in V(G)$, a *color neighbourhood* of v is defined as a set $T \subseteq N(v)$ such that the colors of the edges between v and T are distinct pairwise. A *maximum color neighborhood* $N^c(v)$ of v is a color neighborhood of v with maximum size. And we denote $d^c(v) = |N^c(v)|$ and call it the *color degree* of v.

If $P = v_1v_2\cdots v_p$ is a path, we let $P[v_i, v_j]$ be the subpath $v_iv_{i+1}\cdots v_j$, and $P^-[v_i, v_j] = v_jv_{j-1}\cdots v_i$.

A path or cycle in an edge-colored graph is called *alternating* if any adjacent edges have distinct colors. Besides a number of applications in graph theory and algorithms, the concept of alternating paths and cycles, appears in various other fields: genetics (cf. [8, 9, 10]), social sciences (cf.[7]). A good resource on alternating paths and cycles is the survey paper [2] by J. Bang-Jensen and G. Gutin.

Grossman and Häggkvist[11] were the first to study the problem of the existence of the alternating cycles in c-edge colored graphs. They proved Theorem 1 below in the case $c = 2$. The case $c \geq 3$ was proved by Yeo [14]. Let v be a cut vertex in an edge colored graph G. We say that v *separates colors* if no component of $G - v$ is joined to v by at least two edges of different colors.

Theorem 1 (Grossman and Häggkvist [11], and Yeo [14]). *Let G be an c-edge colored graph, $c \geq 2$, such that every vertex of G is incident with at least two edges of different colors. Then either G has a cut vertex separating colors, or G has an alternating cycle.*

Consider the edge colored complete graph, we use the notation K_n^c to denote a complete graph on n vertices, each edge of which is colored by a color from the set $\{1, 2, \cdots, c\}$. And $\Delta(K_n^c)$ is the maximum number of edges of the same color adjacent to a vertex of K_n^c. And we have the following conjecture due to B. Bollobás and P. Erdős [3].

Conjecture 1 (B. Bollobás and P. Erdős [3]). *If $\Delta(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a Hamiltonian alternating cycle.*

B. Bollobás and P. Erdős managed to prove that $\Delta(K_n^c) < \frac{n}{69}$ implies the existence of a Hamiltonian alternating cycle in K_n^c. This result was improved by C.C. Chen and

D.E. Daykin [6] to $\Delta(K_n^c) < \frac{n}{17}$ and by J. Shearer [13] to $\Delta(K_n^c) < \frac{n}{7}$. So far the best asymptotic estimate was obtained by Alon and Gutin [1].

Theorem 2(Alon and Gutin[1]). *For every $\epsilon > 0$ there exists an $n_o = n_0(\epsilon)$ so that for every $n > n_o$, K_n^c satisfying $\Delta(K_n^c) \leq (1 - \frac{1}{\sqrt{2}} - \epsilon)n$ has a Hamiltonian alternating cycle.*

2 Main results

We study some color degree condition for the existence of the alternating cycles, in particular the shortest alternating cycles and the longest alternating cycles.

We begin with a study of the existence of an alternating cycle with good property. Under color degree conditions, we have

Theorem 3. *Let G be a colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{n+1}{3}$ for every $v \in V(G)$, then G has an alternating cycle AC such that each color in $C(AC)$ appears at most two times in AC.*

Moreover, for the existence of an alternating cycle, we have the following proposition.

Proposition. *For any integer i, there exists a colored graph G_i such that $d^c(v) \geq i$, for every vertex v of G_i, and G_i has no alternating cycles.*

To show the above proposition, we construct the following example by induction.

Let G_1 be an edge e with color $C(e) = 1$. Given G_i, we construct G_{i+1} as follows. First, make $(i+1)$ copies of G_i and denote them by $G_i^1, G_i^2, \cdots, G_i^{i+1}$. Let $\{c_1, c_2, \cdots, c_{i+1}\}$ be the colors such that $\{c_1, c_2, \cdots, c_{i+1}\} \cap C(G_i) = \phi$. Add a new vertex v_{i+1}. For each G_i^j, $1 \leq j \leq i+1$, join v_{i+1} to each vertex of G_i^j, then color these edges with color c_j. Then G_i is a colored graph such that $d^c(v) \geq i$, for every vertex v of G_i, and clearly G_i contains no alternating cycles.

For the shortest alternating cycles, we get result on alternating triangles or alternating quadrilaterals with minimum color degree conditions.

Theorem 4. *Let G be a colored graph with order $n \geq 3$. If $d^c(v) \geq \frac{37n-17}{75}$ for every $v \in V(G)$, then G contains at least one alternating triangle or one alternating quadrilateral.*

We also give a bound for the longest alternating cycles.

Theorem 5. *Let G be a colored graph with order n. If $d^c(v) \geq d \geq \frac{n}{2}$, for every vertex of $v \in V(G)$, then G has an alternating cycle with length at least $\lceil \frac{d}{2} \rceil + 1$.*

In fact, we think that the bound in Theorem 5 is not sharp, and we propose the following conjecture.

Conjecture 2. *Let G be a colored graph with order n. If $d^c(v) \geq \frac{n}{2}$, for every vertex of $v \in V(G)$, then G has a Hamiltonian alternating cycle.*

We have the following example to show that if the above conjecture is true, it would be best possible. For any integer m, let K_m, K'_{m+1} be two edge-proper-colored complete graphs with order $m, m+1$, respectively. For every vertex $u \in K_m$ and every vertex $u' \in K'_{m+1}$, add the edges uu' and let $C(uu') = c_0$, where $c_0 \notin C(K_m) \cup C(K'_{m+1})$. The new colored graph is denoted by B. Clearly, $|V(B)| = n = 2m+1$. Moreover for every vertex v of B, it holds that $d^c(v) \geq m = \frac{n-1}{2}$, and B contains no Hamiltonian alternating cycle.

The proofs of the main results in Theorem $3, 4, 5$ will be given in Section 3.

3 Proofs of the main results

Proof of Theorem 4.

By contradiction. Suppose that G is a colored graph such that $d^c(v) \geq \frac{37n-17}{75}$ for every vertex v of G, and G contains neither alternating triangles nor alternating quadrilaterals.

For an edge uv, let $N_1^c(u), N_1^c(v)$ denote a maximum color neighborhood of u, v, respectively, such that $v \in N_1^c(u)$ and $u \in N_1^c(v)$. Let $N^c(u,v)$ denote $N_1^c(u) \cup N_1^c(v)$ such $|N_1^c(u) \cup N_1^c(v)|$ is maximum. And choose an edge $uv \in E(G)$ such that $|N^c(u,v)|$ is maximum.

Assume that $N_1^c(u) = \{v, u_1, u_2, \cdots, u_s\}$ and $N_1^c(v) \backslash N_1^c(u) = \{u, v_1, v_2, \cdots, v_t\}$, in which $s = d^c(u) - 1$. Let $X = \{u_1, \cdots, u_s, v_1, \cdots, v_t\}$. Note that $|N^c(u,v)| = s + t + 2$. Consider the graph $G[X]$, and we have the following lemma.

Lemma 1.1. *Suppose $e \in E(G[X])$, then the following hold:*
(i) If $e = u_iu_j (1 \leq i, j \leq s)$, then $C(e) \in \{C(uu_i), C(uu_j)\}$.
(ii) If $e = v_iv_j (1 \leq i, j \leq t)$, then $C(e) \in \{C(vv_i), C(vv_j)\}$.
(iii) If $e = u_iv_j (1 \leq i \leq s, 1 \leq j \leq t)$ and $C(uu_i) \neq C(vv_j)$, then $C(e) \in \{C(uu_i), C(vv_j)\}$.

Proof. Clearly (i) and (ii) hold, otherwise we can obtain an alternating triangle, which gets a contradiction.

If (iii) does not hold, then there exists an edge $e = u_iv_j$ $(1 \leq i \leq s, 1 \leq j \leq t)$ such that $C(uu_i) \neq C(vv_j)$ and $C(e) \notin \{C(uu_i), C(vv_j)\}$. Since $v, u_i \in N_1^c(u)$, then $C(uu_i) \neq C(uv)$. Similarly, we obtain that $C(vv_j) \neq C(uv)$. Then we can get an alternating quadrilateral : uvv_ju_iu, a contradiction. □

Construct a digraph as follows.

(1). In graph $G[X]$, do the following operation: deleting the edges $e = u_iv_j$ if $C(uu_i) = C(vv_j)$, $1 \leq i \leq s$ and $1 \leq j \leq t$. (Note that if $C(uu_i) = C(vv_j)$ and $u_iv_j \in E(G[X])$, then $C(u_iv_j) = C(uu_i) = C(vv_j)$). After the operation, the graph is named $G_1[X]$.

(2). Then give an orientation of $G_1[X]$: For an edge $xy \in E(G_1[X])$, if $C(xy) = C(uy)$ or $C(xy) = C(vy)$, then the orientation of xy is from x to y. Otherwise, by Lemma 1.1, $C(xy) = C(ux)$ or $C(xy) = C(vx)$, then the orientation of xy is from y to x.

After the orientation, the digraph is denoted by D_1. For any vertex $w \in V(D_1)$, let $N^+_{D_1}(w)$ denote the outneighbors of w in D_1 and $d^+_{D_1}(w) = |N^+_{D_1}(w)|$. Let $G_0 = G[X \cup \{u, v\}]$.

Lemma 1.2. *If there exists a directed cycle $\overrightarrow{C_p}$ in D_1, then C_p is an alternating cycle in G, moreover each color in $C(C_p)$ appears at most two times in C_p.*

Proof. Firstly, we will prove that C_p is alternating. Assume that xy and yz are adjacent edges of C_p, and furthermore, in $\overrightarrow{C_p}$, the orientations of xy, yz are from x to y, from y to z. By the orientation rule, we conclude that $C(xy) = C(uy)$ or $C(xy) = C(vy)$ and $C(yz) = C(uz)$ or $C(yz) = C(vz)$.

If $C(xy) = C(uy)$ and $C(yz) = C(uz)$ or $C(xy) = C(vy)$ and $C(yz) = C(vz)$, then by the definition of the maximum color neighborhood, it holds that $C(uy) \neq C(uz)$ and $C(vy) \neq C(vz)$, Thus we have that $C(xy) \neq C(yz)$.

Otherwise, without loss of generality, assume that $C(xy) = C(uy)$ and $C(yz) = C(vz)$. Then by (1) and Lemma 1.1(iii), we have that $C(uy) \neq C(vz)$. It follows that $C(xy) \neq C(yz)$.

Thus C_p is an alternating cycle. Moreover by the definition of $N^c(u, v)$, we can conclude that each color in $C(C_p)$ appears at most two times in C_p. □

The *girth* of a digraph D containing directed cycles is the length of the smallest directed cycle in D. Since G has neither alternating triangles nor alternating quadrilaterals, it follows that the girth of D_1 is at least 5.

Lemma 1.3[5]. *Let D be a digraph on m vertices with girth* 5. *Then $\delta^+ < \frac{9(m-1)}{28}$.*

Let $\alpha = \frac{9}{28}$. By Lemma 1.3, there is a vertex w of D_1 such that $d^+_{D_1}(w) < \alpha(|V(D_1)| - 1) = \alpha(s + t - 1) = \alpha(d^c(u) + t - 2)$. Without loss of generality, assume that $w \in N^c_1(u)$. Denote a maximum color neighborhood of w in G_0 by $N^c_{G_0}(w)$. Then it holds that $|N^c_{G_0}(w)| = |N^+_{D_1}(w)| + |v|(or|u|) = d^+_{D_1}(w) + 1$. It follows that

$$|N^c(w)\backslash(X \cup \{u, v\})| \geq d^c(w) - |N^c_{G_0}(w)| > d^c(w) - \alpha(d^c(u) + t - 2) - 1.$$

If $d^c(w) - \alpha(d^c(u) + t - 2) - 1 > t$, then consider the edge uw and it holds that

$$\begin{aligned} |N^c(u, w)| \quad &\geq |\{v, u_1, u_2, \cdots, u_s\}| + |N^c(w)\backslash(X \cup \{u, v\})| + |w| \\ &> s + t + 2 \\ &= |N^c(u, v)|, \end{aligned}$$

a contradiction with the choice of uv.

Then $d^c(w) - \alpha(d^c(u)+t-2) - 1 \le t$, that is $t \ge \frac{d^c(w)}{1+\alpha} - \frac{\alpha d^c(u)}{1+\alpha} + \frac{2\alpha-1}{1+\alpha}$. It follows that

$$\begin{aligned} n \quad & \ge |X| + |u| + |v| + |N^c(w)\backslash(X \cup \{u,v\})| \\ & > d^c(u) + t - 1 + 2 + d^c(w) - \alpha(d^c(u)+t-2) - 1 \\ & \ge \frac{1-\alpha}{1+\alpha} d^c(u) + \frac{2}{1+\alpha} d^c(w) + \frac{5\alpha-1}{1+\alpha}. \end{aligned}$$

Since $d^c(v) \ge \frac{37n-17}{75}$ for every vertex $v \in V(G)$ and $\alpha = \frac{9}{28}$, the above inequality is

$$n > \frac{3-\alpha}{1+\alpha}\frac{37n-17}{75} + \frac{5\alpha-1}{1+\alpha} \ge n.$$

This contradiction completes the proof of Theorem 4. □

Proof of Theorem 3.

We use the same notations and same technique as in the proof of Theorem 4, and omit some details. By contradiction. Suppose that G is a colored graph such that $d^c(v) \ge \frac{n+1}{3}$, for every vertex v of G, and G contains no alternating cycles with the prescribed property.

Similarly, choose an edge $uv \in E(G)$ such that $N^c(u,v)$ is maximum. Assume that $N^c(u,v) = N_1^c(u) \cup N_1^c(v) = X \cup \{u,v\}$. After the deleting and orienting operations in $G[X]$ by the same rule as above, the digraph is denoted by D_1. By Lemma 1.2, there exist no directed cycles in D_1. And we have the following fact.

Fact 2.4. *Every simple m-vertex digraph with minimum out-degree at least 1 has a directed cycle.*

By Fact 2.4, there is a vertex w such that $d^+_{D_1}(w) = 0$. Without loss of generality, assume that $w_1 \in N_1^c(u)$. Let $N^c(w)$ be a maximum color neighbor of w_1 in G, then it holds that$|N^c(w_1)\backslash(X \cup \{u,v\})| \ge d^c(w) - 1$. Then it follows that $d^c(w) - 1 < t$ by the choice of the edge uv. It follows that

$$\begin{aligned} n \quad & \ge |X| + |u| + |v| + |N^c(w)\backslash(x \cup \{u,v\})| \\ & \ge d^c(u) + t - 1 + 2 + d^c(w) - 1 \\ & > d^c(u) + 2d^c(w) - 1 \\ & \ge 3(\frac{n+1}{3}) - 1 = n \end{aligned}$$

This contradiction completes the proof of Theorem 3. □

Proof of Theorem 5.

If $n = 3$, the conclusion holds clearly. So we assume that $n \ge 4$.

By contradiction. Otherwise, let $P = v_1 v_2 \cdots v_l$ be an alternating path of G such that $|P|$ is maximum. Then choose a maximum color neighborhood $N^c(v_1)$ of v_1 such that $v_2 \in N^c(v_1)$. By the maximum of $|P|$, we have $N^c(v_1) \in V(P)$. It follows that $l \geq d+1$, since $|N^c(v_1)| = d^c(v) \geq d$. Choose v_s satisfying the followings:

$\mathbf{R_1}$. $v_s \in N^c(v_1)$.
$\mathbf{R_2}$. $s \geq \lceil \frac{d}{2} \rceil + 1$.
$\mathbf{R_3}$. subject to R_1, R_2, s is minimum.

Since $n \geq 4$ and $d \geq \frac{n}{2}$, we can deduce that $s < l$.

Lemma 3.1. If $v_i \in N^c(v_1)$ and $i \geq s$, then $C(v_i v_{i+1}) \neq C(v_1 v_i)$.

Proof. Otherwise, there exists $i \geq s$ such that $C(v_i v_{i+1}) = C(v_1 v_i)$. Since P is an alternating path, $C(v_{i-1} v_i) \neq C(v_i v_{i+1})$, thus, $P[v_1, v_i] v_i v_1$ is an alternating cycle with length $i \geq s \geq \lceil \frac{d}{2} \rceil + 1$, a contradiction. □

Now choose a maximum color neighborhood of $N^c(v_l)$ of v_l such that $v_{l-1} \in N^c(v_l)$. Similarly, we conclude that $N^c(v_l) \in V(P)$. Then choose t satisfying the followings:

$\mathbf{R_1'}$. $v_t \in N^c(v_l)$.
$\mathbf{R_2'}$. $l - t \geq \lceil \frac{d}{2} \rceil$.
$\mathbf{R_3'}$. subject to R_1', R_2', t is maximum.

Similarly, it holds that $t > 1$. And we have the following lemmas.

Lemma 3.2. If $v_i \in N^c(v_l)$ and $i \leq t$, then $C(v_{i-1} v_i) \neq C(v_i v_l)$.

Proof. Otherwise, as in the proof of Lemma 3.1, we can get an alternating cycle with length at least $\lceil \frac{d}{2} \rceil + 1$, a contradiction. □

Lemma 3.3. $s < t$.

Proof. Otherwise, we have that $s \geq t$. If $s > t$, then $AC^0 = v_1 v_s P[v_s, v_l] v_l v_t P^-[v_t, v_1]$ is an alternating cycle. And $|AC^0| = |P[v_s, v_l]| + |P[v_1, v_t]| \geq 2(d - \lceil \frac{d}{2} \rceil + 1) = 2(\lfloor \frac{d}{2} \rfloor + 1) = 2\lfloor \frac{d}{2} \rfloor + 2 > \lceil \frac{d}{2} \rceil + 1$, a contradiction.

So we assume that $s = t$. If there exists $v_j \in N^c(v_1)$ such that $s + 1 \leq j \leq l - 1$, then there is an alternating cycle $AC^1 = v_1 v_j P[v_j, v_l] v_l v_s P^-[v_s, v_1]$ with length $|AC^1| \geq 2 + |P[v_1, v_s]| \geq 3 + \lceil \frac{d}{2} \rceil$, which gives a contradiction. Similarly, if there exists $v_j \in N^c(v_l)$ such that $2 \leq j \leq s - 1$, we obtain an alternating cycle $v_1 v_s P[v_s, v_l] v_l v_j P^-[v_j, v_1]$ with length $3 + \lceil \frac{d}{2} \rceil$, which also get a contradiction.

Thus we can conclude that $v_j \notin N^c(v_1)$ if $s + 1 \leq j \leq l - 1$ and $v_j \notin N^c(v_l)$ if $2 \leq j \leq s - 1$. On the other hand, by R_3 it holds that $|V(P[v_{s+1}, v_l]) \cap N^c(v_1)| \geq d - \lceil \frac{d}{2} \rceil = \lfloor \frac{d}{2} \rfloor \geq 1$. Clearly $v_l \in N^c(v_1)$. Similarly, we have that $v_1 \in N^c(v_l)$. (Note that it holds that $d = 2, 3$). That is, $C(v_1 v_l) \neq C(v_1 v_2)$ and $C(v_1 v_l) \neq C(v_{l-1} v_l)$. Then

$P[v_1, v_l]v_lv_1$ is an alternating cycle with length at least $l \geq d+1 > \lceil \frac{d}{2} \rceil + 1$, a contradiction. $\square$

Lemma 3.4. For $2 \leq j \leq s-1$, $v_j \notin N^c(v_l)$; And for $t+1 \leq j \leq l-1$, $v_j \notin N^c(v_1)$.

Proof. Without loss of generality, we only prove the first part. Otherwise, there exists $v_j \in N^c(v_l)$ such that $2 \leq j \leq s-1$. Clearly, $j \leq t$, thus by Lemma 3.2 we have that $C(v_{j-1}v_j) \neq C(v_jv_l)$. Then we get an alternating cycle $AC^2 = v_1v_sP[v_s, v_l]v_lv_jP^-[v_j, v_1]$. And it holds that $|AC^2| \geq |P[v_s, v_l]| + 2 \geq \lfloor \frac{d}{2} \rfloor + 2 \geq \lceil \frac{d}{2} \rceil + 1$, a contradiction. $\square$

Denote $N^c(v_1) \cap V(P[v_s, v_t])$, $N^c(v_l) \cap V(P[v_s, v_t])$ by A, B respectively.

Lemma 3.5. $|A| + |B| \geq 2\lfloor \frac{d}{2} \rfloor + 1$.

Proof. By R_1, $|N^c(v_1) \cap V(P[v_s, v_l])| \geq d - (|P[v_1, v_{s-1}]| - 1) \geq d - (\lceil \frac{d}{2} \rceil - 1) = \lfloor \frac{d}{2} \rfloor + 1$. Then by Lemma 3.4, we obtain that $N^c(v_1) \cap V(P[v_s, v_l]) = N^c(v_1) \cap (V(P[v_s, v_t]) \cup \{v_l\}) = A \cup (N^c(v_1) \cap \{v_l\})$. It follows that $|A| \geq \lfloor \frac{d}{2} \rfloor + 1 - |N^c(v_1) \cap \{v_l\}|$. Similarly, we can obtain that $|B| \geq \lfloor \frac{d}{2} \rfloor + 1 - |N^c(v_l) \cap \{v_1\}|$. Then $|A| + |B| \geq 2\lfloor \frac{d}{2} \rfloor + 2 - (|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}|)$.

If $|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}| = 2$, this means that $v_l \in N^c(v_1)$ and $v_1 \in N^c(v_l)$. Thus, by the definition of a maximum color neighborhood, it holds that $C(v_lv_l) \neq C(v_1v_2)$ and $C(v_1v_l) \neq C(v_{l-1}v_l)$. Then $P[v_1, v_l]v_lv_1$ is an alternating cycle with length $l \geq d+1 > \lceil \frac{d}{2} \rceil + 1$, a contradiction. Thus it holds that $|N^c(v_1) \cap \{v_l\}| + |N^c(v_l) \cap \{v_1\}| \leq 1$, then $|A| + |B| \geq 2\lfloor \frac{d}{2} \rfloor + 1$. $\square$

Now we completes the proof as follows. We have that $|V(P[v_s, v_t])| \leq n - |V(P[v_1, v_{s-1}])| - |V(P[v_{t+1}, v_l])| \leq n - \lceil \frac{d}{2} \rceil - \lceil \frac{d}{2} \rceil \leq 2d - 2\lceil \frac{d}{2} \rceil \leq 2\lfloor \frac{d}{2} \rfloor$. And by Lemma 3.5, $|N^c(v_1) \cap V(P[v_s, v_t])| + |N^c(v_l) \cap V(P[v_s, v_t])| = |A| + |B| \geq 2\lfloor \frac{d}{2} \rfloor + 1$, then it follows that there exists v_j $(s+1 \leq j \leq t)$ such that $v_j \in N^c(v_1)$ and $v_{j-1} \in N^c(v_l)$. So we get an alternating cycle $v_1v_jP[v_j, v_l]v_lv_{j-1}P^-[v_{j-1}, v_1]$ with length $l \geq |P[v_1, v_s]| \geq l \geq d+1 \geq \lceil \frac{d}{2} \rceil + 1$, a contradiction. This completes the proof.

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