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A DEGREE CONDITION IMPLYING THAT EVERY MATCHING IS CONTAINED IN A HAMILTONIAN CYCLE

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A degree condition implying that every matching is contained in a hamiltonian cycle

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Abstract

We give a degree sum condition for three independent vertices under which every matching of a graph lies in a hamiltonian cycle. We can show that the bound for the degree sum is almost best possible.

Résumé

Nous obtenons une condition portant sur la somme des degrés de trois sommets indépendants pour que tout couplage d'un graphe soit contenu dans un cycle hamiltonien. Nous prouvons que la borne obtenue sur la somme des degrés est presque la meilleure possible.

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1 Introduction

Let G be a graph, $\mathrm{V}(G)$ denotes the vertex set of G and $\mathrm{E}(G)$ denotes the edge set of G. By $\mathrm{d}(x)$ or $\mathrm{d}_G(x)$ we denote *the degree of the vertex x in the graph G.*

In 1960 O. Ore [8] proved the following:

Theorem 1 *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n,$$

then G is hamiltonian.

Later many Ore type theorems dealing with degree-sum conditions were proved.

In particular J.A. Bondy [2] proved:

Theorem 2 *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If for any independent vertices $x, y, z \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant \frac{3n-2}{2},$$

then G is hamiltonian.

Let G be a graph and let $k \geqslant 1$. We shall call a set of k independent edges a *k-matching* or simply a *matching.* Sometimes the number of edges of a k-matching M we will denote by $|M|$.

About cycles through matchings in general graphs K.A. Berman proved in [1] the following result conjectured by R. Häggkvist in [6].

Theorem 3 *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n + 1,$$

then every matching lies in a cycle.

Theorem 3 has been improved by B. Jackson and N.C. Wormald in [7]. R. Häggkvist [6] gave also a sufficient condition for a general graph to contain any matching in a hamiltonian cycle. We give this theorem below in a slightly improved version obtained in [10] by A.P. Wojda.

Let $\mathcal{G}_\backslash$ be the family of graphs $G = \overline{K}_{\frac{n+2}{3}} * H$, where H is any graph of order $\frac{2n-3}{3}$ containing a perfect matching if $\frac{n+2}{3}$ is an integer, and $\mathcal{G}_n = \emptyset$ otherwise ($*$ denotes the join of graphs).

Theorem 4 *Let G be a graph on $n \geqslant 3$ vertices. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant \frac{4n-4}{3},$$

then every matching of G lies in a hamiltonian cycle, unless $G \in \mathcal{G}_n$.

M. Las Vergnas [9] have proved a similar result, but the bound for degree sum depends on the number of edges of the matching M.

Theorem 5 *Let G be a graph on $n \geqslant 3$ vertices and let k be an integer $0 \leqslant k \leqslant \frac{n}{2}$. If for any pair of independent vertices $x, y \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant n + k,$$

then every k-matching of G lies in a hamiltonian cycle.

We have tried to find new conditions dealing with degree sum of three independent vertices under which every matching from a graph G is contained in a hamiltonian cycle.

First we have obtained the following extension theorem:

Theorem 6 *Let G be a 3-connected graph on $n \geqslant 3$ vertices such that for any independent vertices $x, y, z \in \mathrm{V}(G)$, we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n. \tag{1}$$

Let M be a matching in G. If there exists a cycle of G containing M, then there exists a hamiltonian cycle of G containing M.

Theorem 6 shows that if a graph G satisfies (1) and a matching of G lies in a cycle, then this cycle can be extended to a hamiltonian cycle.

Using Theorem 6 we prove the following analog of Theorem 2, about hamiltonian cycles through matchings:

Theorem 7 *Let G be a 3-connected graph on $n \geqslant 3$ vertices and let M be a matching in G such that for any independent vertices $x, y, z \in \mathrm{V}(G)$ we have:*

$$\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n, \tag{2}$$

then there exists a hamiltonian cycle containing every edge of M or G has a minimal odd M-edge cut-set.

A minimal odd M-edge cut-set is a subset of M such that its suppression disconnects the graph G and which has no proper subset being an H-edge cut-set.

Theorem 7 is an analog of Theorem 2, about hamiltonian cycles through matchings.

Note that the bound $2n$ in Theorem 7 is almost best possible. Let $p \geqslant 2$ and consider a complete graph K_{2p} with a perfect p-matching. We define the graph $G = (p+1)K_1 * K_{2p}$, ($*$ denotes the join of graphs). In this graph $n = 3p+1$ and G is 3-connected. For any independent $x, y, z \in \mathrm{V}(G)$ we have $\mathrm{d}(x) + \mathrm{d}(y) + \mathrm{d}(z) \geqslant 2n-2$ and there is no hamiltonian cycle containing the p-matching from K_{2p}. So the bound $2n$ is almost best possible.

Let G be a graph. We define $\alpha(G)$, *the stability number of* G, as the cardinal of a maximum independent set of vertices of G.

From Theorem 7 we have the following Corollary:

Corollary 8 *Let G be a 3-connected graph on $n \geqslant 6$ vertices and let M be a matching of G. If $\alpha(G) = 2$, then there is a hamiltonian cycle of G containing M or G has a minimal odd M-edge cut-set.*

2 Notation and preliminary results

For any $A \subset G$ and $x \in \mathrm{V}(G)$, we denote by $\mathrm{N}_A(x)$ *the set of all neighbors of the vertex x in A*. Note that A can be a subgraph or a set of vertices. For $\mathrm{N}_G(x)$ we will sometimes write shortly $\mathrm{N}(x)$.

We will only use oriented cycles and paths. Let C be a cycle with a given orientation and $x \in \mathrm{V}(C)$, then x^- is *the predecessor of* x and x^+ is its *successor* according to the orientation of C. For any subest $A \subset \mathrm{V}(C)$ we denote by A^+ the set of successors of vertices from A and by A^- the set of predecessors of vertices from A.

Let $C : c_1 \dots c_l$ be a cycle (or a path) in G with a given orientation. For any pair of vertices c_i, $c_j \in \mathrm{V}(C)$ with $i < j$ we can define four intervals:

- $]c_i, c_j[$ is the path $c_{i+1} \dots c_{j-1}$.
- $[c_i, c_j[$ is the path $c_i \dots c_{j-1}$.
- $]c_i, c_j]$ is the path $c_{i+1} \dots c_j$.
- $[c_i, c_j]$ is the path $c_i \dots c_j$.

Observe that these four intervals are subgraphs of the cycle (or the path) C.

Let u and v be two vertices of a graph G. We shall define $\epsilon(uv) : \epsilon(uv) = 1$ if $uv \in \mathrm{E}(G)$ and $\epsilon(uv) = 0$ if $uv \notin \mathrm{E}(G)$.

Let W be a property defined for all graphs of order n and let k be a non-negative integer. The property W is said to be k-stable if whenever $G + xy$ has property W and $\mathrm{d}_G(x) + \mathrm{d}_G(y) \geqslant k$ then G itself has property W.

Let $k, s_1, \dots s_l$ be positive integers. We call S *a path system of length* k if the components of S are paths:

$$\begin{array}{ll} P_1 : & x_0^1 x_1^1 \dots x_{s_1}^1, \\ & \vdots \\ P_l : & x_0^l x_1^l \dots x_{s_l}^l \end{array}$$

and $\sum_{i=1}^{l} s_i = k$.

Note that a k-matching is a path system of length k.

J.A. Bondy and V. Chvátal [3] proved the following theorem, which we shall need in the proof:

Theorem 9 *Let n and k be positive integers with $k \leqslant n - 3$. Then the property of being k-edge-hamiltonian is $(n + k)$-stable.*

For a matching M, we denote by $\mathrm{V}(M)$ the set of all end vertices of the edges from M.

For notation and terminology not defined above a good reference should be [4].

3 Proof of Theorem 6

Let $k = |M|$ and let C be a longest cycle of G containing every edge of M. We assume that C is not hamiltonian. We denote by $R = \mathrm{V}(G) \setminus \mathrm{V}(C)$ the set of vertices of G not in C. Let $u \in R$. Since G is 3-connected, we have $P_1[u, a]$, $P_2[u, b]$, $P_3[u, c]$ three internally disjoint paths from u to C, where $a, b, c \in \mathrm{V}(C)$. If at least two edges between a^-a, b^-b, c^-c are edges of the matching M, at least two between aa^+, bb^+, cc^+ are not in M. Without loss of generality we may assume that $aa^+ \notin M$, $bb^+ \notin M$.

The three vertices u, a^+, b^+ are independent, so from (1) we have:

$$\mathrm{d}(u) + \mathrm{d}(a^+) + \mathrm{d}(b^+) \geqslant 2n. \tag{3}$$

3.1 Neighbors of u, a^+, b^+ in R and C

Since the vertices a^+, b^+ and u don't have common neighbors in R and are independent, we have: $\mathrm{d}_R(a^+) + \mathrm{d}_R(b^+) + \mathrm{d}_R(u) \leqslant |\mathrm{V}(R)| - 1$.

As C is a longest cycle containing M, if $x \in \mathrm{V}(C)$ is a neighbor of u and x^+ is a neighbor of a^+ or b^+, then $xx^+ \in M$ and hence

$$\big(\mathrm{N}_C(u)\big)^+ \cap [\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)] \subset \{\alpha \in \mathrm{V}(C),\ \alpha^-\alpha \in M\}$$

and

$$\left|(\mathrm{N}_C(u))^+ \cap [\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)]\right| \leqslant k.$$

As $\left|\mathrm{N}_C(u)^+ \cup \mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)\right| \leqslant |\mathrm{V}(C)|$, we have:

$$|\mathrm{N}_C(u)| + |\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)| \leqslant |\mathrm{V}(C)| + k.$$

Moreover

$$\left|\mathrm{N}_C(a^+) \cup \mathrm{N}_C(b^+)\right| = \left|\mathrm{N}_C(a^+)\right| + \left|\mathrm{N}_C(b^+)\right| - \left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right|.$$

To find an upper bound for $\left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)\right|$ we shall study vertices of $\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

Let $C_1 = C[a,b]$ and $C_2 = C[b,a]$ be the two intervals on the cycle with endvertices a and b. Let $x \in C_1,\ x \in \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

If $xx^+ \notin M$ and $x^+ \in \mathrm{N}_C(a^+)$, then the cycle:

$$P_1[u,a]a^- \dots b^+ x x^- \dots a^+ x^+ \dots P_2[b,u]$$

is a cycle containing M longer then C, a contradiction.

Hence $x^+ \notin \mathrm{N}_C(a^+)$.

Similarly if $x^-x \notin M$ then $x^- \notin \mathrm{N}_C(b^+)$.

In both cases $x^+ \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+),\ x^- \notin \mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

Using similar arguments for the interval C_2, we have no two consecutive vertices of $C \setminus \mathrm{V}(M)$ in the set $\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+)$.

3.2 Relations on degrees of $a^+,\ b^+,\ u$

If we consider any path P_i of C between two edges of M, we have:

$$\left|\mathrm{N}_C(a^+) \cap \mathrm{N}_C(b^+) \cap \mathrm{V}(P_i)\right| \leqslant \left\lceil \frac{|\mathrm{V}(P_i)|}{2} \right\rceil.$$

Let for $i \geqslant 2,\ n_i$ be the cardinality of the set of the paths on C of length $i-1$, between two edges of M. The following relations must be satisfied:

$$\begin{aligned} |\mathrm{V}(C)| &= \sum_{i\geqslant 2} i n_i \\ |\mathrm{V}(M)| &= \sum_{i\geqslant 2} n_i \\ \left|\mathrm{N}_C(a^+)\cap \mathrm{N}_C(b^+)\right| &\leqslant \sum_{i\geqslant 2} \left\lceil \frac{i}{2}\right\rceil n_i\,. \end{aligned}$$

As

$$\mathrm{d}_C(a^+)+\mathrm{d}_C(b^+)+\mathrm{d}_C(u) \leqslant |\mathrm{V}(C)|+k+\left|\mathrm{N}_C(a^+)\cap \mathrm{N}_C(b^+)\right|,$$

we have:

$$\begin{aligned} \mathrm{d}_C(a^+)+\mathrm{d}_C(b^+)+\mathrm{d}_C(u) &\leqslant \sum_{j\geqslant 1}(2jn_{2j}+(2j+1)n_{2j+1})+ \\ &\sum_{j\geqslant 1}(n_{2j}+n_{2j+1})+\sum_{j\geqslant 1}(jn_{2j}+(j+1)n_{2j+1}) \leqslant \\ &\sum_{j\geqslant 1}(3j+1)n_{2j}+\sum_{j\geqslant 1}(3j+3)n_{2j+1} \leqslant \\ &\sum_{j\geqslant 1}4jn_{2j}+\sum_{j\geqslant 1}(4j+2)n_{2j+1}. \end{aligned}$$

Hence $\mathrm{d}_C(a^+)+\mathrm{d}_C(b^+)+\mathrm{d}_C(u) \leqslant 2\,|\mathrm{V}(C)|$

and

$\mathrm{d}(a^+)+\mathrm{d}(b^+)+\mathrm{d}(u) \leqslant 2\,|\mathrm{V}(C)|+|\mathrm{V}(R)|-1 \leqslant$

$2(|\mathrm{V}(C)|+|\mathrm{V}(R)|)-|\mathrm{V}(R)|-1 = 2n-|\mathrm{V}(R)|-1,$

a contradiction with (3).

This contradiction ends the proof of Theorem 6.

□

4 Proof of Theorem 7

Let $k=|M|$.

4.1 Preliminary Remarks

Remark 1 *For two independent vertices x, $y \in \mathrm{V}(G)$ two cases can occur:*

1. *If there exists a vertex z such that x, y, z are independent, then $\mathrm{d}(x) + \mathrm{d}(y) \geqslant 2n - \mathrm{d}(z) \geqslant n + 3$.*
2. *If there is no vertex in G independent with x and y, then $\mathrm{N}(x) \cup \mathrm{N}(y) \cup \{x, y\}$ covers $\mathrm{V}(G)$ and $\mathrm{d}(x) + \mathrm{d}(y) \geqslant n - 2$.*

Remark 2 *If x and y are independent vertices satisfying $\mathrm{d}(x)+\mathrm{d}(y) = n-2+\epsilon$, with $0 \leqslant \epsilon \leqslant 3$ we are in the second case. We may assume $\mathrm{d}(y) \leqslant \mathrm{d}(x)$. If u_1 and u_2 are independent vertices in $\mathrm{N}(x)\setminus\mathrm{N}(y)$, then $\mathrm{d}(u_1)+\mathrm{d}(u_2) \geqslant 2n-\mathrm{d}(y) \geqslant \frac{3n-1}{2} = n + \frac{n-1}{2}$. If n is even, then $\mathrm{d}(u_1) + \mathrm{d}(u_2) \geqslant n + \frac{n}{2} \geqslant n + k$. If n is odd, then any matching of G has at most $\frac{n-1}{2}$ edges, then we have again $\mathrm{d}(u_1) + \mathrm{d}(u_2) \geqslant n + k$. In any case u_1u_2 is in the $(n + k)$-closure of G. From Theorem 9 we can assume that $\mathrm{N}(x) \setminus \mathrm{N}(y)$ is a complete graph.*

4.2 Definition of a θ-graph through a matching in the graph G

Definition 1 *A θ-graph through a matching M is the union of two cycles C_1 and C_2 whose intersection is a path of length at least one and such that $M \subset \mathrm{E}(C_1) \cup \mathrm{E}(C_2)$ and every edge of M incident with a vertex of $C_1 \cap C_2$ lies in $C_1 \cap C_2$.*

This notion has been introduced by Berman [1].

4.3 Strategy of the proof

We will prove the theorem by contradiction. We assume that for a matching M there is no hamiltonian cycle containing M. We consider a cycle C in G which satisfies the following conditions:

1. $|\mathrm{E}(C) \cap M|$ is maximum.
2. Up to condition (1) the length of C is maximum, so by Theorem 6, C is a hamiltonian cycle.

Existence of a θ-graph

Let $M' = \mathrm{E}(C) \cap M$. By assumption $M' \neq M$ and then there exists an edge $e = xy \in M$, $e \notin \mathrm{E}(C)$. The edge $e = xy$ is a chord of the hamiltonian cycle. Let $C_1 = xx^+ \dots yx$ and $C_2 = xx^- \dots yx$. Note that $(C_1 \cup C_2)$ satisfies the definition of a θ-graph through $M' \cup \{e\}$.

Maximality conditions for a θ-graph

Let $\Gamma(C_1, C_2)$ be a θ-graph through $M' \cup \{e\}$ satisfying moreover:

1. The intersection $C_1 \cap C_2$ is maximum.
2. Under condition (1) $|\mathrm{V}(\Gamma(C_1, C_2))|$ is maximum.

In $\Gamma(C_1, C_2)$, we denote by $P, Q, R',\ R$ the paths defined respectively by: $R' = C_1 \cap C_2 = xr_1r_2...r_\gamma y,\ R = r_1r_2...r_\gamma,\ P = C_1 \setminus C_2 = p_1p_2...p_\alpha$ with $xp_1 \in \mathrm{E}(C_1)$, $Q = C_2 \setminus C_1 = q_1q_2...q_\beta$ with $xq_1 \in \mathrm{E}(C_2)$.

Sometimes we will write Γ instead of $\Gamma(C_1, C_2)$.

Inequalities and consequences

Remark 3 *The edges $xp_1,\ xq_1, yp_\alpha, yq_\beta$ are not in M, then p_1 and q_β are independent and and q_1 and p_α are independent.*

Remark 4 *We can apply the same arguments as Berman [1] (see inequalities (4) — (12) in [1]) and we have the following inequality:*

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) + \mathrm{d}(p_\alpha) + \mathrm{d}(q_\beta) \leqslant 2n.$$

Since the graph G satisfies the condition (2) (i.e. for any independent vertices $w_1, w_2, w_3 \in \mathrm{V}(G)$ we have $\mathrm{d}(w_1) + \mathrm{d}(w_2) + \mathrm{d}(w_3) \geqslant 2n$) and by Remark 1 we have the following inequalities:

$$\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \geqslant n - 2,$$
$$\mathrm{d}(q_1) + \mathrm{d}(p_\alpha) \geqslant n - 2.$$

Hence:

$$\mathrm{d}(q_1) + \mathrm{d}(p_\alpha) \leqslant n + 2,$$
$$\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \leqslant n + 2$$

and there is no vertex independent of p_1 and q_β and no vertex independent of q_1 and p_α.

Remark 5 *Without loss of generality we may assume that $\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \leqslant n,\ \mathrm{d}(q_\beta) \leqslant \dfrac{n}{2}$ and by Remark 2, $\mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ is a complete graph.*

4.3.1 Basic Lemmas

The following lemmas involve the neighbors of the vertices p_1, q_1, p_α, and q_β on the paths R, P, Q:

Lemma 1

1. *If uv is an edge of R not in M, then two cases can occur:*
   (a) *Vertices p_1 and q_1 are both adjacent to u and v and p_α and q_β are independent of u and v and even there is no path internally disjoint with Γ, from u and v to p_α and q_β.*
   (b) *Vertices p_α and q_β are both adjacent to u and v and vertices p_1 and q_1 are independent of u and v and even there is no path internally disjoint with Γ, from u or v to p_1 or q_1.*
2. *Consequently for any $r \in \mathrm{V}(R)$ we have two possibilities:*
   (a) *Vertices p_1 and q_1 are both adjacent to r and p_α and vertices q_β are independent of r.*
   (b) *Vertices p_α and q_β are both adjacent to r and vertices p_1 and q_1 are independent of r.*
3. *If $xr_1 \notin M$, then r_1p_1, $r_1q_1 \in \mathrm{E}(G)$ and r_1p_α, $r_1q_\beta \notin \mathrm{E}(G)$ and respectively if $yr_\gamma \notin M$, then $r_\gamma p_\alpha$, $r_\gamma q_\beta \in \mathrm{E}(G)$ and $r_\gamma p_1$, $r_\gamma q_1 \notin \mathrm{E}(G)$.*

Proof of Lemma 1:

We shall prove first 1. As $\mathrm{N}(p_1) \cup \mathrm{N}(q_\beta) = \mathrm{V}(G) \setminus \{p_1, q_\beta\}$ and $\mathrm{N}(q_1) \cup \mathrm{N}(p_\alpha) = \mathrm{V}(G) \setminus \{q_1, p_\alpha\}$, the vertex u is adjacent to at least one of the vertices p_1 or q_β. The assumption of the proof is that no cycle contains every edge of $M \cap \mathrm{E}(\Gamma)$. If we assume $up_1 \in \mathrm{E}(G)$, then $p_\alpha v \notin \mathrm{E}(G)$ and $q_\beta v \notin \mathrm{E}(G)$, that implies $q_1 v \in \mathrm{E}(G)$ and $p_1 v \in \mathrm{E}(G)$. Hence $q_\beta u \notin \mathrm{E}(G)$ and $p_\alpha v \notin \mathrm{E}(G)$, that implies $q_1 v \in \mathrm{E}(G)$. If we assume $up_1 \notin \mathrm{E}(G)$, then $q_\beta u \in \mathrm{E}(G)$, that implies $q_1 v \notin \mathrm{E}(G)$, then $p_\alpha v \in \mathrm{E}(G)$ that implies $q_1 u \notin \mathrm{E}(G)$, then $p_\alpha u \in \mathrm{E}(G)$ that implies $p_1 v \notin \mathrm{E}(G)$ then $q_\beta v \in \mathrm{E}(G)$. Moreover we can replace the condition $wt \notin \mathrm{E}(G)$ by no path from w to t, internally disjoint of Γ exists, where w may be u or v, and t may be p_1, p_α, q_1, q_β.

Using similar arguments we can show 2 and 3.

□

Note that from Lemma 1 we have $\mathrm{d}_R(p_1) = \mathrm{d}_R(q_1)$ and similarly $\mathrm{d}_R(p_\alpha) = \mathrm{d}_R(q_\beta)$.

Lemma 2 *If p_ip_{i+1} is an edge from* $\mathrm{E}(P) \setminus M$*, then* $q_\beta p_{i+1} \notin \mathrm{E}(G)$, $q_1p_i \notin \mathrm{E}(G)$, $q_\beta p_i \notin \mathrm{E}(G)$ $q_1p_{i+1} \notin \mathrm{E}(G)$ *and* p_1p_i, p_1p_{i+1}, $p_\alpha p_i$, $p_\alpha p_{i+1}$ *are edges of* G*. Similarly if q_iq_{i+1} is an edge from* $\mathrm{E}(Q) \setminus M$*, then* $p_1q_i \notin \mathrm{E}(G)$, $p_\alpha q_{i+1} \notin \mathrm{E}(G)$, $p_1q_{i+1} \notin \mathrm{E}(G)$, $p_\alpha q_i \notin \mathrm{E}(G)$ *and* q_1q_i, q_1q_{i+1}, $q_\beta q_i$, $q_\beta q_{i+1}$ *are edges of* G.

Proof of Lemma 2:

The hypothesis of maximality of $C_1 \cap C_2$ implies that the edges q_1p_i, $q_\beta p_{i+1}$, p_1q_i, $p_\alpha q_{i+1}$ are not in $\mathrm{E}(G)$. As $\mathrm{N}(p_1) \cup \mathrm{N}(q_\beta) \cup \{p_1, q_\beta\}$ or $\mathrm{N}(q_1) \cup \mathrm{N}(p_\alpha) \cup \{q_1, p_\alpha\}$ cover $\mathrm{V}(G)$ the edges p_1p_{i+1}, $p_\alpha p_i$, q_1q_{i+1}, $q_\beta q_i$ are in $\mathrm{E}(G)$. If $p_1p_{i+1} \in \mathrm{E}(G)$, $q_\beta p_i \notin \mathrm{E}(G)$ elsewhere

$$xr_1...r_\gamma yp_\alpha...p_{i+1}p_1p_2...p_iq_\beta...q_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction. Hence $p_1p_i \in \mathrm{E}(G)$.

The proofs for the other vertices are similar.

□

4.3.2 Steps of the proof

We will first study the case where $\alpha = \beta = 2$ and obtain the existence of a minimal odd M-edge cut-set. Then we will assume that $\alpha \geqslant 3$ or $\beta \geqslant 3$, we will use the structure of the neighborhood of the vertices p_1, q_1, p_α, q_β and obtain a contradiction.

4.4 Proof of Theorem 7 for $\alpha = \beta = 2$.

4.4.1 Claims and Corollaries

Let $S = G \setminus \Gamma$.

Claim 1 *The vertex p_1 has no neighbor in S.*

Proof of Claim 1:

Assume that $w \in \mathrm{V}(S)$ is adjacent to p_1. Let $\pi[w, t]$, with $t \in \mathrm{V}(\Gamma)$, denote a path from $p_1$to Γ, internally disjoint of Γ. First $t \neq q_2$ elsewhere we obtain a cycle through $M' \cup \{e\}$. Because of the maximality of $|\mathrm{V}(\Gamma)|$, $t \neq x$. If $t = q_1$, $xq_2 \notin \mathrm{E}(G)$, then $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, $w \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, then $wx \in \mathrm{E}(G)$, a contradiction with the hypothesis of maximality of $|\mathrm{V}(\Gamma)|$. Then $wq_1 \notin \mathrm{E}(G)$, and $wp_2 \in \mathrm{E}(G)$. We can deduce that $t \neq y$ because of the maximality of $|\mathrm{V}(\Gamma)|$.

At last, as G is 3-connected, there exists an other path (than the edges wp_1 and wp_2), say $\pi[w, r]$ from w to Γ, with $r \in \mathrm{V}(R)$. At least one of the edges rr^+ or r^-r is not in M, r^+ in the first case, r^- in the second case is adjacent to p_1 or p_2, a contradiction with Lemma 1.

□

Claim 2 *The edge p_2q_2 is in* $\mathrm{E}(G)$.

Proof of Claim 2:

Case 1: $p_1q_1 \in \mathrm{E}(G)$ **or there exists a path** $\pi[p_1, q_1]$ **internally disjoint with** Γ.

Then $xp_2 \notin \mathrm{E}(G),\ xq_2 \notin \mathrm{E}(G)$ elsewhere we obtain a cycle through $M' \cup \{e\}$. The conditions $x \in \mathrm{N}(p_1)\backslash\mathrm{N}(q_2),\ xp_2 \notin \mathrm{E}(G)$ imply $p_2 \in \mathrm{N}(q_2)$ i.e. $p_2q_2 \in \mathrm{E}(G)$.

Case 2: $p_1q_1 \notin \mathrm{E}(G)$ **and there exists no path** $\pi[p_1, q_1]$ **internally disjoint with** Γ.

We assume $p_2q_2 \notin \mathrm{E}(G)$. Then $p_2 \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$. We have: $\mathrm{N}(p_1) \subset \mathrm{V}(R) \cup \{x, y, p_2\}$.

Let $r \in \mathrm{V}(R)$ be a neighbor of p_1. We have $r \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2),\ p_2 \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$, that implies $rp_2 \in \mathrm{E}(G)$, a contradiction with Lemma 1. So $\mathrm{N}_R(p_1) = \emptyset$, and $\mathrm{N}(p_1) \subset \{x, y, p_2\}$.

Since G is 3-connected $\mathrm{N}(p_1) = \{x, y, p_2\}$. The condition $\mathrm{d}(p_1) \geqslant \mathrm{d}(q_2)$ implies that $|V(R)| \leqslant 1$.

If $R = \emptyset$, it is easy to see that xy is a minimal odd M-edge cut-set, a contradiction.

If $R = \{r_1\}$ and $yr_1 \in M$, then C' : $xp_1p_2r_1yq_2q_1x$ is a cycle through $M' \cup \{e\}$, a contradiction.

If $R = \{r_1\}$ and $xr_1 \in M$, then C' : $xr_1p_2p_1yq_2q_1x$ is a cycle through $M' \cup \{e\}$, a contradiction and Claim 2 is proved.

□

Note that we have also the following Corollaries from Claim 2:

Corollary 1 *Both pairs of vertices $\{y\,, p_1\}$ and $\{y\,, q_1\}$ are independent and have no common neighbors in S.*

Corollary 2 *If vertices $\{y\,, p_1\}$ (or $\{y\,, q_1\}$) have no common neighbors on R, then $p_1q_1 \in \mathrm{E}(G)$ and y is adjacent to every neighbor of p_2 (or q_2) on R.*

Proof of Corollary 2:
If there exists a set of three independent vertices containing y and p_1 (or q_1), then $\mathrm{d}(y) + \mathrm{d}(p_1) \geqslant n + 3$. Note that we have: $\mathrm{N}(p_1) \cap \mathrm{N}(y) \subset \mathrm{V}(R) \cup \{x, p_2\}$, $|\mathrm{N}_R(p_1) \cap \mathrm{N}_R(y)| \geqslant 3$.

Hence, if $\mathrm{N}_R(p_1) \cap \mathrm{N}_R(y) = \emptyset$, then there is no independent set of three vertices containing p_1 and y, and $p_1q_1 \in \mathrm{E}(G)$. As $\mathrm{N}_R(y) \cup \mathrm{N}_R(p_1) = \mathrm{V}(R)$, by Lemma 1, y is adjacent to every vertex of $\mathrm{N}_R(p_2) = \mathrm{N}_R(q_2)$.

□

4.4.2 Proof of the Theorem 7

We recall that we consider the case $\alpha = \beta = 2$.

By Lemma 1 the sets $\mathrm{N}_R(p_1) = \mathrm{N}_R(q_1)$ and $\mathrm{N}_R(p_2) = \mathrm{N}_R(q_2)$ define a partition of the set of the vertices of R and by Remark 2 we may assume that $\mathrm{N}_R(p_1)$ is a complete graph. If an edge $ab \in \mathrm{E}(R)$ is such that a is adjacent to p_1 (and q_1) and b is adjacent to p_2 (and q_2), then by Lemma 1 $ab \in M$. Let $\{e_j = a_jb_j,\ a_j \in \mathrm{N}_R(p_1),\ b_j \in \mathrm{N}_R(p_2)\}$ be the set of these edges. The path R can be partitioned into subpaths: $R_0 = R[x, a_1](= \{x\}$ if $a_1 = x)$, $R_1 = R[b_1...b_2]$, ... $R_s = R[b_s, y](= \{y\}$ if $b_s = y)$. Every vertex of $R_0, R_2, \ldots, R_{2j} \ldots$ is adjacent to p_1 (and q_1), and every vertex of $R_1, R_3, \ldots, R_s$ is adjacent to p_2 (and q_2). Note that s is odd. If no other edge exists between $\mathrm{N}(p_1) \cup \{p_1, q_1\}$ and $\mathrm{N}(p_2) \cup \{p_2, q_2\}$, then the set

$$\{e_j = a_jb_j,\ a_j \in \mathrm{N}(p_1),\ b_j \in \mathrm{N}(p_2),\ 1 \leqslant j \leqslant s\} \cup \{p_1, p_2\} \cup \{q_1, q_2\}$$

is an odd minimal M-edge cut-set.

Otherwise there exists an edge $cd \in \mathrm{E}(G)$, with $c \in \mathrm{N}(p_1),\ d \in \mathrm{N}(p_2)$.

Case 1: There is an edge r_ty, with $r_t \in \mathrm{N}_R(p_1)$

Subcase 1.1

If $r_tr_{t+1} \notin M$, then

$$xr_1...r_tyr_\gamma...r_{t+1}q_1q_2p_2p_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2

If $r_tr_{t+1} \in M,\ r_{t+1} \in \mathrm{N}_R(p_1)$, then $r_{t-1} \in \mathrm{N}_R(p_1),\ r_{t+2} \in \mathrm{N}_R(p_1)$ and $r_{t-1}r_{t+2} \in \mathrm{E}(G)$.

In this case

$$xr_1...r_{t-1}r_{t+2...}r_\gamma yr_tr_{t+1}q_1q_2p_2p_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.3

If $r_tr_{t+1} \in M$, $r_{t+1} \in \mathrm{N}_R(p_2)$, then

$$xr_1...r_{t-1}p_1p_2r_{t+2}...r_\gamma yr_tr_{t+1}q_2q_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Case 2: **The vertex y is not adjacent to any vertex of** $\mathrm{N}_R(p_1)$.

By Corollary 2, y is adjacent to any vertex of $\mathrm{N}_R(p_2)$. Let $c = r_t \in \mathrm{N}_R(p_1),\ d = r_m \in \mathrm{N}_R(p_2)$ such that $cd \in \mathrm{E}(G)$.

Subcase 2.1 : $r_t r_{t+1}, r_m r_{m+1} \notin M$ **or** $r_{t-1} r_t, r_{m-1} r_m \notin M$.

If $t < m$, then

$$xr_1...r_t r_m r_{m-1}...r_{t+1} q_1 q_2 r_{m+1}...y p_2 p_1 x$$

or

$$xr_1...r_{t-1} q_1 q_2 y r_\gamma ... r_m r_t r_{t+1}...r_{m-1} p_2 p_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

If $t > m$, then

$$xr_1...r_m r_t r_{t-1}...r_{m+1} q_2 q_1 r_{t+1}...r_\gamma y p_2 p_1 x$$

or

$$xr_1...r_{m-1} q_2 q_1 r_{t-1}...r_m r_t...y p_2 p_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.2 : $r_t r_{t+1} \in M$ **and** $r_{m-1} r_m \in M$ **if** $t < m$, $r_{t-1} r_t \in M$ **and** $r_m r_{m+1} \in M$ **if** $t > m$.

There exists i, i between t and m, such that $r_i r_{i+1} \notin M$. The vertices r_i and r_{i+1} are both adjacent to p_1 and q_1 or to p_2 and q_2.

Subcase 2.2.1 : **The vertices r_i and r_{i+1} are both adjacent to p_1 and q_1.**

If $t < m$, then since r_{t-1}, $r_{i+1} \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_2)$ we have $r_{t-1} r_{i+1} \in \mathrm{E}(G)$ and

$$xr_1...r_{t-1} r_{i+1}...r_m r_t r_{t+1}...r_i q_1 q_2 r_{m+1}...y p_2 p_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

If $t > m$, then

$$xr_1...r_{m-1} q_2 q_1 r_i r_{i-1}...r_m r_t...r_{i+1} r_{t+1}...y p_2 p_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.2.2 : **The vertices r_i and r_{i+1} are both adjacent to p_2 and q_2.**

In this case r_i and r_{i+1} are adjacent to y.

If $t < m$, then

$$xr_1...r_{t-1}q_1q_2r_ir_{i-1}...r_tr_mr_{m-1}...r_{i+1}yr_\gamma...r_{m+1}p_2p_1x)$$

is a cycle through $M' \cup \{e\}$, a contradiction.

If $t > m$, then

$$(xr_1...r_{m-1}p_2p_1r_{t+1}...yr_{i+1}...r_tr_m...r_iq_2q_1x)$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.3 : *$r_{t-1}r_t \in M$ and $r_mr_{m+1} \in M$* **if** *$t < m$* **or** *$r_tr_{t+1} \in M$ and $r_{m-1}r_m \in M$* **if** *$t > m$.*

So if $t < m$, $r_{m-1}y \in \mathrm{E}(G)$, if $t > m$, $r_{m+1}y \in \mathrm{E}(G)$.

Hence if $t < m$, then

$$xr_1...r_tr_m...yr_{m-1}...r_{t+1}q_1q_2p_2p_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

If $t > m$,

$$xr_1...r_mr_tr_{t+1}yr_{m+1}...r_{t-1}q_1q_2p_2p_1x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

The proof of Theorem 7 for $\alpha = \beta = 2$ is complete.

□

4.5 Proof of Theorem 7 for $\alpha \geqslant 3$ or $\beta \geqslant 3$

Case 1 : $p_1q_1 \in \mathrm{E}(G)$

Remark 6 *: The hypothesis of maximality of the intersection $C_1 \cap C_2$ implies that the edges p_1p_2 and q_1q_2 are in M.*

Remark 7 *: $xq_\beta \notin \mathrm{E}(G)$, $xp_\alpha \notin \mathrm{E}(G)$ and there is no path $\pi[xq_\beta]$ or $\pi[xp_\alpha]$ internally disjoint of Γ, elsewhere we obtain a cyle through $M' \cup \{e\}$.*

Remark 8 *: Since $x \in \mathrm{N}(p_1)\backslash\mathrm{N}(q_\beta)$, $p_\alpha \in \mathrm{N}(p_1)\cup\mathrm{N}(q_\beta)$ and $xp_\alpha \notin \mathrm{E}(G)$, $p_\alpha \notin \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, we have $p_\alpha q_\beta \in \mathrm{E}(G)$, that implies $p_\alpha p_{\alpha-1} \in M$, $q_\beta q_{\beta-1} \in M$ and $yp_1 \notin \mathrm{E}(G)$, $yq_1 \notin \mathrm{E}(G)$.*

Remark 9 *: If $w \in \mathrm{N}_S(p_1)$ and $w \notin \mathrm{N}(q_\beta)$, then $w \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, that implies $wx \in \mathrm{E}(G)$, a contradiction with the hypothesis of maximality of $|\mathrm{V}(\Gamma)|$. Hence $\mathrm{N}_S(p_1) = \emptyset$.*

By Lemma 2 and the property that p_1p_2, q_1q_2, $p_\alpha p_{\alpha-1}$, $q_{\beta-1}q_\beta$ are in M, we deduce the following Lemma:

Lemma 3

1. *The vertex p_1 is independent of $q_2, \ ,..., q_\beta$ and adjacent to $p_2, \ ..., p_{\alpha-1}$.*
2. *The vertex q_1 is independent of $p_2, \ ..., \ p_\alpha$ and adjacent to $q_2, \ ..., \ q_{\beta-1}$.*
3. *The vertex p_α is independent of $q_1, \ ..., \ q_{\beta-1}$ and adjacent to $p_2, \ ..., \ p_{\alpha-1}$.*
4. *The vertex q_β is independent of $p_1, \ ..., \ p_{\alpha-1}$ and adjacent to $q_2, \ ..., \ q_{\beta-1}$.*

We recall that we consider the case $\alpha \geqslant 3$, $\beta \geqslant 3$.

Subcase 1.1 : $\alpha \geqslant 3$

By Lemma 2, $p_{\alpha-1} \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$. As $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, $xp_{\alpha-1} \in \mathrm{E}(G)$, the edges p_1p_2 and $p_{\alpha-1}p_\alpha$ are in M, then the condition $\alpha > 2$ implies $\alpha \geqslant 4$. By Lemma 3 $p_{\alpha-2}p_\alpha \in \mathrm{E}(G)$, and then

$$xr_1...r_\gamma y q_\beta ... q_1 p_1 p_2 ... p_{\alpha-2} p_\alpha p_{\alpha-1} x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2 : $\alpha = 2, \ \beta \geqslant 3$

The vertex p_2 is a common neighbor of p_1 and q_β, then $\mathrm{d}(p_1) + \mathrm{d}(q_\beta) \geqslant n-1$ that implies $\mathrm{d}(q_1) + \mathrm{d}(p_2) \leqslant n+1$ and $\min\{\mathrm{d}(q_1), \mathrm{d}(p_2)\} \leqslant \dfrac{n+1}{2}$.

So, in case $\mathrm{d}(q_1) \geqslant \mathrm{d}(p_2)$ the $(n+k)$-closure of $\mathrm{N}(q_1) \setminus \mathrm{N}(p_2)$ is a complete graph, elsewhere $\mathrm{d}(p_2) \geqslant \mathrm{d}(q_1)$. We shall examine both cases.

Subcase 1.2.1 : $\mathrm{d}(q_1) \geqslant \mathrm{d}(p_2)$

In this Case $\mathrm{N}(q_1) \setminus \mathrm{N}(p_2)$ is a complete graph. As $\beta \geqslant 3$, $q_2q_3 \notin M$, $q_3 \in \mathrm{N}(q_1) \setminus \mathrm{N}(p_2)$ and $xq_3 \in \mathrm{E}(G)$. Then $(x...yp_2p_1q_1q_2q_\beta...q_3x)$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 1.2.2 : $\mathrm{d}(p_2) \geqslant \mathrm{d}(q_1)$.

The following inequalities are satisfied:

$$\begin{array}{rcl} \mathrm{d}(p_1) & \geqslant & \mathrm{d}(q_\beta), \\ \mathrm{d}(p_2) & \geqslant & \mathrm{d}(q_1), \\ \mathrm{d}(p_2)+\mathrm{d}(q_1) & \geqslant & \mathrm{d}(p_1)+\mathrm{d}(q_\beta). \end{array}$$

They imply that:

$$\mathrm{d}(p_1)+\mathrm{d}(p_2) \geqslant n-1.$$

We have: $\mathrm{N}(p_1) = \mathrm{N}_R(p_1) \cup \{p_2, x, q_1\}$ and $\mathrm{N}(p_2) = \mathrm{N}_R(p_2) \cup \{p_1, y, q_\beta\} \cup \mathrm{N}_S(p_1)$

By Lemma 1 $\mathrm{d}_R(p_1) + \mathrm{d}_R(p_2) = |\mathrm{V}(R)| = \gamma$. $\mathrm{d}(p_1) + \mathrm{d}(p_2) = \mathrm{d}_R(p_1) + \mathrm{d}_R(p_2) + 6 + \mathrm{d}_S(p_1) + \mathrm{d}_S(p_2) \leqslant \gamma + 6 + |\mathrm{V}(S)|$. Since $n = \gamma + \beta + 4 + |\mathrm{V}(S)|$, we obtain:

$n-1 = \gamma+\beta+4+|\mathrm{V}(S)|-1 \leqslant \mathrm{d}(p_1)+\mathrm{d}(p_2) \leqslant \gamma+6+|\mathrm{V}(S)|$ that implies $\beta \leqslant 3$. We have $q_1q_2 \in M$ and $q_{\beta-1}q_\beta \in M$, then if $\beta \leqslant 3$, $q_1q_2 = q_{\beta-1}q_\beta$, $\beta = 2$, a contradiction.

Case 2: $p_1q_1 \notin \mathrm{E}(G)$

Lemma 4

1. *The vertex p_1 is independent of $q_1,\ q_2, ..., q_\beta$ and adjacent to $p_2,\ ..., p_\alpha$.*
2. *The vertex q_1 is independent of $p_1,\ ...,\ p_\alpha$ and adjacent to $q_2,\ ...,\ q_\beta$.*
3. *The vertex p_α is independent of $q_1,\ ...,\ q_{\beta-1}$ and adjacent to $p_1,\ ...,\ p_{\alpha-1}$.*
4. *The vertex q_β is independent of $p_1,\ ...,\ p_{\alpha-1}$ and adjacent to $q_1,\ ...,\ q_{\beta-1}$.*

Proof of Lemma 4:

The condition $q_1 \notin \mathrm{N}(p_1)$ implies that $q_1 \in \mathrm{N}(q_\beta)$, the condition $p_1 \notin \mathrm{N}(q_1)$ implies that $p_1 \in \mathrm{N}(p_\alpha)$ i.e. the edges p_1p_α and q_1q_β are in $\mathrm{E}(G)$. Let i be a minimal integer such that $p_1q_i \in \mathrm{E}(G)$. For $1 \leqslant j \leqslant i-1$, $p_1q_j \notin \mathrm{E}(G)$, then $q_\beta q_j \in \mathrm{E}(G)$. The hypothesis of maximality of $C_1 \cap C_2$ implies that $q_iq_{i+1} \in M$ and then $q_{i-1}q_i \notin M$.

The cycle

$$(xr_1...r_\gamma yp_\alpha...p_1q_i...q_\beta q_{i-1}...q_1x)$$

is a cycle through $M' \cup \{e\}$, a contradiction.

The vertex p_1 is independent of $q_1,\ q_2, ..., q_\beta$, hence q_β is adjacent to $q_1,\ q_2, ..., q_{\beta-1}$.

The proofs for the other vertices are similar.

□

Subcase 2.1 : $p_\alpha q_\beta \notin \mathrm{E}(G)$

Claim 3 *If* $p_\alpha q_\beta \notin \mathrm{E}(G)$*, then* $\mathrm{N}_R(p_1) = \mathrm{N}_R(q_1) = \emptyset$.

Proof of Claim 3:

If $p_\alpha \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ and $u \in \mathrm{N}_R(p_1)$, then $uq_\beta \notin \mathrm{E}(G)$, $u \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ then $u \in \mathrm{N}_R(p_1) \cap \mathrm{N}_R(p_\alpha)$, a contradiction with Lemma 1.

□

Claim 4 *At least one of the edges* xp_α *or* xq_β *is in* $\mathrm{E}(G)$.

Proof of Claim 4:

If $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, x is adjacent to every vertex of $\mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$, then $xp_\alpha \in \mathrm{E}(G)$.

□

Corollary 3 $\mathrm{N}_S(p_1) \cap \mathrm{N}_S(q_1) = \emptyset$.

Claim 5 *At least one of the edges* yp_1 *or* yq_1 *is in* $\mathrm{E}(G)$.

Proof of Claim 5:

Vertices p_1 and q_1 have no common neighbor in S. The following inequality is satisfied:

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant \alpha + \beta + |\mathrm{V}(S)| + \epsilon(yp_1) + \epsilon(yq_1)$$

and since

$$n = \alpha + \beta + \gamma + 2 + |\mathrm{V}(S)|,$$

we have:

$$\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant n.$$

The vertices p_1 and q_1 are not in any set of three independent vertices and so Claim 5 is proved.

□

Subcase 2.1.1 : $\gamma = |\mathrm{V}(R)| = 0$.

In this case $xy \in M$. As G is 3-connected, $G \setminus \{x, y\}$ is connected. The conditions $\epsilon(xp_\alpha) + \epsilon(xq_\beta) \geqslant 1$, $\epsilon(yp_1) + \epsilon(yq_1) \geqslant 1$ imply that there is no path $\pi[p_1, q_1]$, $\pi[p_1, q_\beta]$, $\pi[p_\alpha, q_1]$, $\pi[p_\alpha, q_\beta]$ elsewhere there is a cycle through $M' \cup \{e\}$. As G is $3-$connected, there exists a path $\pi[p_i, q_j]$, with $2 \leqslant i \leqslant \alpha - 1$, $2 \leqslant j \leqslant \beta - 1$. We can easily construct a cycle through $M' \cup \{e\}$.

Subcase 2.1.2 : $\gamma \geqslant 1$, $\mathrm{d}(q_1) \geqslant \mathrm{d}(p_\alpha)$.

By Claims 4 and 5, $\epsilon(xp_\alpha) + \epsilon(xq_\beta) \geqslant 1$ and $\epsilon(yp_1) + \epsilon(yq_1) \geqslant 1$, then:

$\mathrm{d}(p_1) + \mathrm{d}(q_1) + \mathrm{d}(p_\alpha) + \mathrm{d}(q_\beta) \geqslant 2n - 2$ that implies $\mathrm{d}(p_1) + \mathrm{d}(q_1) \geqslant n - 1$ We have $\mathrm{N}(p_1) \subset \{x, y\} \cup \{p_2 ..., p_\alpha\} \cup S$, $\mathrm{N}(q_1) \subset \{x, y\} \cup \{q_2 ..., q_\beta\} \cup S$, $\mathrm{N}_S(p_1) \cap \mathrm{N}_S(q_1) = \emptyset$ and so $\mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant \alpha + \beta + |\mathrm{V}(S)| + \epsilon(yp_1) + \epsilon(yq_1)$. Moreover $n = \alpha + \beta + \gamma + 2 + |\mathrm{V}(S)|$.

The inequality $\mathrm{d}(p_1) + \mathrm{d}(q_1) \geqslant n - 1$ gives: $\gamma + 1 \leqslant \epsilon(yp_1) + \epsilon(yq_1)$. Hence $\gamma = 1 = \epsilon(yp_1) = \epsilon(yq_1)$. If $xr_1 \in M$, $(xr_1q_\beta ... q_1yp_\alpha ... p_1x)$ is a cycle through $M' \cup \{e\}$, if $r_1y \in M$, $(xp_1 ... p_\alpha r_1yq_\beta ... q_1x)$ is a cycle through $M' \cup \{e\}$, a contradiction.

Subcase 2.1.3 : $\gamma \geqslant 1$, $\mathrm{d}(p_\alpha) > \mathrm{d}(q_1)$

Note that $\mathrm{d}(p_\alpha) + \mathrm{d}(q_1) = \alpha + \beta + \gamma + |\mathrm{V}(S)| + \epsilon(yq_1) + \epsilon(xp_\alpha) \leqslant n$. Hence the $(n + k)$-closure of $\mathrm{N}(p_\alpha) \setminus \mathrm{N}(q_1)$ is a complete graph. Let $u \in \mathrm{N}_R(p_\alpha)$, $u \in \mathrm{N}(p_\alpha) \setminus \mathrm{N}(q_1)$ and $p_1 \in \mathrm{N}(p_\alpha) \setminus \mathrm{N}(q_1)$. This imply $up_1 \in \mathrm{E}(G)$, a contradiction with Lemma 1. Hence $\mathrm{N}_R(p_1) = \mathrm{N}_R(p_\alpha) = \emptyset$, $\gamma = 0$, a contradiction with the hypothesis of Subcase 2.1.3.

Subcase 2.2 : $p_\alpha q_\beta \in \mathrm{E}(G)$

Claim 6 *If* $p_\alpha q_\beta \in \mathrm{E}(G)$, $yp_1 \notin \mathrm{E}(G)$, $yq_1 \notin \mathrm{E}(G)$.

By Claim 6 $\mathrm{d}(p_1) + \mathrm{d}(q_\beta) = \alpha + \beta + \gamma + \epsilon(xq_\beta) + 1 + |\mathrm{V}(S)| = n - 1 + \epsilon(xq_\beta) \leqslant n$. $\mathrm{d}(q_1) + \mathrm{d}(p_\alpha) = \alpha + \beta + \gamma + \epsilon(xp_\alpha) + 1 + |\mathrm{V}(S)| = n - 1 + \epsilon(xp_\alpha) \leqslant n$.

Subcase 2.2.1 : $\mathrm{d}(q_1) \geqslant \mathrm{d}(p_\alpha)$

The $(n + k)$-closure of $\mathrm{N}(q_1) \setminus \mathrm{N}(p_\alpha)$ is a complete graph, we may assume that $\mathrm{N}(q_1) \setminus \mathrm{N}(p_\alpha)$ is complete. Vertices p_1, q_1, y are independent and thus $\mathrm{d}(p_1) + \mathrm{d}(q_1) \geqslant n + 3$. Recall that $\mathrm{d}_R(q_1) = \mathrm{d}_R(p_1)$.

The inequalities:

$$\begin{aligned} \mathrm{d}(p_1) &= \alpha + \mathrm{d}_R(p_1) + \mathrm{d}_S(p_1) \\ \mathrm{d}(q_1) &= \beta + \mathrm{d}_R(q_1) + \mathrm{d}_S(q_1) \end{aligned}$$

imply that:

$$\alpha + \beta + 2\mathrm{d}_R(p_1) + \mathrm{d}_S(p_1) + \mathrm{d}_S(q_1) \geqslant n + 3.$$

If $xp_\alpha \in \mathrm{E}(G)$ or $xq_\beta \in \mathrm{E}(G)$, $\mathrm{N}_S(p_1) \cap \mathrm{N}_S(q_1) = \emptyset$.

If $xp_\alpha \notin \mathrm{E}(G)$ and $xq_\beta \notin \mathrm{E}(G)$, $x \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$.; if $w \in \mathrm{N}_S(p_1)$, $w \in \mathrm{N}(p_1) \setminus \mathrm{N}(q_\beta)$ and $xw \in \mathrm{E}(G)$ a contradiction with the hypothesis of maximality of $|\mathrm{V}(\Gamma)|$.

Then $\mathrm{d}_S(p_1) + \mathrm{d}_S(q_1) \leqslant |\mathrm{V}(S)|$.

Note that $n+3 \leqslant \mathrm{d}(p_1) + \mathrm{d}(q_1) \leqslant \alpha + \beta + |\mathrm{V}(S)| + 2\mathrm{d}_R(p_1)$, this imply that $\alpha + \beta + \gamma + |\mathrm{V}(S)| + 5 \leqslant \alpha + \beta + |\mathrm{V}(S)| + 2\mathrm{d}_R(p_1)$. Since $2\mathrm{d}_R(p_1) \geqslant \gamma + 5$ we have $\mathrm{d}_R(p_1) \geqslant 5$.

If $\alpha > 2$, $p_{\alpha-1}p_\alpha \in M$, $p_{\alpha-2}p_{\alpha-1} \notin M$.

Let $r_i r_{i+1}$ be an edge of R not in M, with r_i and r_{i+1} adjacent to p_1. Vertices r_i and r_{i+1} are adjacent to $p_{\alpha-1}$ and $p_{\alpha-2}$.

$$xr_1 ... r_i p_{\alpha-2} ... p_1 p_\alpha p_{\alpha-1} r_{i+1} ... r_\gamma y q_\beta ... q_1 x$$

is a cycle through $M' \cup \{e\}$, a contradiction.

If $\beta > 2$, the argument is similar.

Subcase 2.2.2 : $\mathrm{d}(p_\alpha) > \mathrm{d}(q_1)$

$y \in \mathrm{N}(p_\alpha) \setminus \mathrm{N}(q_1)$, $p_1 \in \mathrm{N}(p_\alpha) \setminus \mathrm{N}(q_1)$, then $yp_1 \in \mathrm{E}(G)$, a contradiction with Claim 6.

The proof of Theorem 7 is complete.

□

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