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ABOUT b-COLOURING OF REGULAR GRAPHS

EL SAHILI A / KOUIDER M

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

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CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

About b-colouring of regular graphs

Amine El Sahili and Mekkia Kouider

Abstract. Is the b-chromatic number of a d-regular graph of girth 5 equal to $d+1$? We study this problem by giving some partial answers.

Keywords: graph colouring, b-chromatic number, girth

1. Introduction

A b-coloring of a graph G is a proper coloring of the vertices of G such that there exists a vertex in each color class joined to at least a vertex in each other color class, such a vertex is called a dominating vertex. The b-chromatic number of a graph G, denoted by $b(G)$, is the maximal integer k such that G may have a b-coloring by k colors. This parameter has been defined by Irving and Manlove [**2**]. They proved that determining $b(G)$ for an arbitrary graph G is an NP-complete problem.

For a given graph G, it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G) + 1$.

In [**5**]) Hoang and Kouider characterize all bipartite graphs G and all P_4-sparse graphs G such that each induced subgraph H of G satisfies $b(H) = \chi(H)$, where $\chi(H)$ is the chromatic number of H. They also prove that every $2K_2$-free and $\overline{P_5}$-free graph G has $b(G) = \chi(G)$.

An important problem is to characterize those graphs G such that $b(G) = \Delta(G) + 1$. If we are limited to regular graphs, Kratochvil et al. proved in [**3**] that for a d-regular graph G with at least d^4 vertices, $b(G) = d + 1$. In [**4**] one of us proved that for every graph G with girth at least 6, $b(G)$ is at least the minimum degree of the graph, and if this graph is d-regular then $b(G) = d + 1$.

Two examples show that the result is not extendable to every regular graph. The simpler one is the cycle C_4, we have $b(C_4) = 2 < 3$. An other example containing triangles is the graph G consisting of two

triangles $x_1x_2x_3$ and $y_1y_2y_3$ such that x_iy_i is an edge $1 \leq i \leq 3$. It is not difficult to show that $b(G) < 4$. If cycles of order less than or equal 4 are not allowed, then we are leaded to study regular graphs of girth 5, the subject of this note.
By putting a supplementary condition, we do a step in the hoped direction.

THEOREM 1. *Let G be a d-regular graph with girth 5 and containing no cycles of order 6. Then the b-chromatic number of G is $d+1$.*

PROPOSITION 1. *Let G be a d-regular graph. If $V(G)$ can be decomposed into $d+1$ stables $S_1, S_2, \cdots, S_{d+1}$ such that for each i, j there is a perfect matching between S_i and S_j, then $b(G) = d+1$.*

By forbidding P_7 we get a lower bound for arbitrary graphs.

PROPOSITION 2. *For a P_7-free graph G of girth 5 we have $b(G) > \frac{\delta - 3}{4}$ where δ is minimal degree of G.*

2. Proof of Theorem1

The following proposition on regular graphs of girth greater than 5 was proved in [**4**].

PROPOSITION 3. *Any d-regular graph with girth 6 has a b-chromatic number equal to $d+1$*

PROOF. Consider a vertex v and its d neighbors $v_1, v_2, \cdots, v_d$. We start our coloring by giving v the color $d+1$ and each vertex v_i the color i. v is then a dominating vertex. Note that no neighbor of v_s other than v is equal nor joined to a neighbor of v_t where $1 \leq s < t \leq d$. Then neighborhoods of the vertices v_i can be colored in such a way that v_i becomes a dominating vertex for all i, $1 \leq i \leq d$. We may easily complete to obtain a b-coloring of G by $d+1$ colors □

LEMMA 1. *Let f be a non constant mapping from E into F where E and F are two finite sets such that $|E| = |F| \geq 2$. Then there is a bijection g from E to F such that $f(x) \neq g(x)$ for all x in E.*

PROOF. We argue by induction on the cardinal of the set E. If $|E| = 2$, then a non constant mapping is a bijection. Simply g will be the other possible bijection from E to F. Suppose that the property holds for n and let E be a set containing $n+1$ elements. Set $E = \{x_1, x_2, \cdots, x_{n+1}\}$, $F = \{y_1, y_2, \cdots, y_{n+1}\}$. If f is a bijection, say for example that $f(x_i) = y_i$, $1 \leq i \leq n+1$. Then the bijection g defined by $f(x_i) = y_{i+1}$, $1 \leq i \leq n$, and $f(x_{n+1}) = y_1$, is a hoped bijection. Otherwise, there is an element in F which is not in $f(E)$. We may suppose that y_{n+1} is such a point. Consider the restriction of f on $E' = \{x_1, x_2, \cdots, x_n\}$, it is a mapping from E' to $F' = \{y_1, y_2, \cdots, y_n\}$. If this mapping is constant, that is there exists $s \in [1, n]$ such that $f(x_i) = y_s$, $1 \leq i \leq n$. then $f(x_{n+1}) \neq y_s$ since f is not constant on E. Consider any bijection g from E' to $F \backslash \{y_s\}$ and extend g to E by putting $g(x_{n+1}) = y_s$. Otherwise we apply induction to get a bijection g from E' to F' verifying $f(x_i) \neq g(x_i), 1 \leq i \leq n$, We extend g to E by putting $g(x_{n+1}) = y_{n+1}$. A convenient bijection is then constructed. □

Proof of the theorem. The cases $d = 1, 2$ are easily checked. So we prove the theorem for $d \geq 3$. Consider a vertex v and its d neighbors $v_1, v_2, \cdots, v_d$. We start our coloring by giving v the color $d+1$ and each vertex v_i the color i. The vertex v is then a dominating vertex. Now we will color the neighborhoods of the vertices v_i in such a way that v_i becomes a dominating vertex for all i, $1 \leq i \leq d$. The $d-1$ neighbors of v_1 other than v are taken colors $2, \cdots, d$. Suppose that all the neighbors of $v_1, \cdots, v_{k-1}$, $k-1 < d$ are colored such that v_i is a dominating vertex for all i, $1 \leq i \leq k-1$, and let us color the neighbors of v_k. First we remark that no colored vertex is a neighbor of v_k other than v; and, no two distinct colored vertices, different from v, are joined to the same vertex in the neighborhood of v_k since in all these cases, we have either a cycle of order less than 5 or a cycle of order 6. Let E be the set of all the neighbors of v_k other than v and let F be the set of the colors i such that $1 \leq i \leq d$ and $i \neq k$. We define from E to F the mapping as follows:

If $u \in E$ is joined to a vertex of color $i \in F$, then put $f(u) = i$. We give arbitrary images of non used colors in F to the other vertices in such a way that f is not a constant mapping. It will be not so even if all the vertices in E are joined to colored one. In fact, if $f(u) = s \neq k$ for all $u \in E$, then... $k = d$ and v_i has a neighbor of color s for all $i \in \{1, \cdots, d-1\}$ since two distinct neighbors of v_i have always two distinct colors. In particular v_s has a neighbor of color s, a contradiction. Hence may apply the lemma to construct a bijection g from E to F such that $f(u) \neq g(u)$ for all $u \in E$. We color the vertices in E by putting $c(u) = g(u)$ for all $u \in E$. Once all the neighbors of v_i are colored we complete by giving to each other vertex a convenient color. It may be easily verified that the obtained coloring is a b-coloring. So $b(G) = d+1$ □

3. Proof of Proposition 2

Proof. Suppose to the contrary that $b(G) = m \leq \dfrac{\delta - 3}{4}$ and consider an $m-$coloration with m dominating vertices $x_1, x_2, \cdots, x_m$. Set $X = \{x_1, x_2, \cdots, x_m\}$. For all $1 \leq t \leq m, t \neq i$ let y_{it} be a vertex of color t in $N(x_i)$. Set $Y_t = \{y_{it}, 1 \leq i \leq m\}$. We will introduce a new color $m+1$. For every $x \in V(G)$, we define the following sets: $S_1(x)$ is a maximal subset of $N(x)$ containing $N(x) \cap X$ in which each two vertices have different colors, $S_2(x) = \{y \in N(x), (N(y) - \{x\}) \cap X \neq \phi\}$,

$S(x,t) = \{y \in N(x), N(y) \cap Y_t \neq \phi\}$. First remark that, as the girth is at least 5, each one of these sets contains at most m elements. Since $m \leq \dfrac{\delta - 3}{4}$, then x_1 has a neighbor which is not in any one of those sets relative to x_1, we give the color $m+1$ to this neighbor. By supposing that at a step $i \leq m$ the set $N(x_j)$ $j < i$ contains exactly one vertex of color $m+1$, we color a neighbor of x_i as we have done for x_1. We get an $(m+1)$-coloring in which $x_i, 1 \leq i \leq m$, is a dominating vertex such that $N(x_i)$ contains exactly one vertex of color $m+1$. Let $S_3(x) = \{y \in N(x), N(y)$ contains a vertex of color $m+1\}$, We consider a vertex x of color $m+1$ such that $|S_1(x)|$ is maximum. If $|S_1(x)| = m$ then we get a dominating vertex of the color $m+1$; this contradicts the definition of $b(G)$. Then we suppose that there is a color t not used by a vertex in $S_1(x)$, since $m \leq \dfrac{\delta - 3}{4}$ then x has at least 3 neighbors y_1, y_2 and y_3 which are not in any one of the sets $S_i(x)$ or $S(x,t)$. The vertex y_1 is joined to a vertex z of color t since otherwise we give to it the color t which is a contradiction. Suppose that the vertex z is not joined to a vertex of color $m+1$. If we have a missing color $j \neq m+1$ in $N(z) - \{y_1\}$, then we give the color j to z and similarly we may change the color of all the neighbors of y_1 having the color t and we are leaded to the first case. Else we give the color $m+1$ to z, t to y_1. We get a dominating vertex of the color $m+1$, a contradiction. Then z is joined to a vertex w of color $m+1$. By definition, the vertex w is joined to a dominating vertex x_i. Let $j \in \{2, 3\}$ be such that $x_i y_j \notin E(G)$. It can be easily verified that there is no edge between two non consecutive vertices on the path $y_j x y_1 z w x_i y_{it}$. Thus G contains an induced path P_7. □

4. Proof of Proposition 1

An other parameter of graphs closed to the b-coloring which is deeply investigated is the chromatic number of the square graph, see for instance [**1**] and [**6**]. Given a graph G, the square of G is the graph G^2 obtained by adding edges to G between any two vertices of G of distance 2. Clearly we may verify that $\chi(G^2) \geq \Delta(G) + 1$. If we have the equality in the case of a d-regular graph, we get obviously $b(G) = d + 1$. We will establish this equality under some particular conditions. First we give the following characterization of a d-regular graph G with $\chi(G^2) = d + 1$.

PROPOSITION 4. *Let G be a d-regular graph. Then $\chi(G^2) = d+1$ if and only if $V(G)$ can be decomposed into $d+1$ stables $S_1, S_2, \cdots, S_{d+1}$ such that for each i, j there is a perfect matching between S_i and S_j.*

PROOF. For the necessary condition, consider a $(d + 1)$-coloring of G^2 and let $S_1, S_2, \cdots, S_{d+1}$ be the stables defined be the colors. For any two of them, say S_i and S_j, it is sufficient to remark that $|N(x_i) \cap S_j| = |N(x_j) \cap S_i| = 1$ for all $x_i \in S_i$ and $x_j \in S_j$. For the sufficient condition, give color i to vertices in S_i, $1 \leq i \leq d+1$. we have $|N(x_i) \cap S_j| \geq 1$ for all $i \neq j$ with $x_i \in S_i$. If $|N(x_i) \cap S_j| \geq 2$ then $d(x_i) > d + 1$, a contradiction. We get a $(d + 1)$-coloring of G^2. □

The proposition 1 is then a corollary.

References

[1] N. Alon and B. Mohar, The chromatic number of graph power, Combinatorics Probability and Computing 11 (1993), 1-10.
[2] I. W. Irving and D. F. Manlove, The b-chromatic number of a graph, Discrete Applied Math.,91 (1999), 127-141.
[3] J. Kratochvil, Zs Tuza, and M. Voigt, On the b-chromatic number of graphs, Lectures Notes in Computer Science, Springer, Berlin, 2573 (2002), 310-320.
[4] M.Kouider,b-chromatic number of a graph, subgraphs and degrees Rapport interne LRI 1392.
[5] C.T. Hoang, Kouider, M., On the b-dominating coloring of graphs, Discrete Applied Maths, 152 (2005) no.1-3, 176-186
[6] K. W. Lih, W. F. Wang, X. Zhu, Coloring the square of a K_4-minor free graph, Discrete Math., 269 (2003), 303-309