On the complementation orbits of graphs

François Genest*
L.R.I., U.M.R. 8623, Bât. 490, Université Paris-Sud, 91405 Orsay, France†
genesfra@iro.umontreal.ca

Résumé

La complémentation locale d'un graphe simple $G = (V, E)$ au sommet $u \in V$ inverse la relation d'adjacence de G sur l'ensemble des voisins de u. Étant donné un ensemble fini V, la complémentation locale induit une action de groupe sur l'ensemble des graphes simples ayant V comme ensemble de sommets. Chaque orbite induite par cette action est appelée une *orbite de Kotzig.* Les membres d'une orbite de Kotzig donnée peuvent être représentés par des suites finies de complémentations locales, une par classe d'équivalence obtenue en quotientant par le *stabilisateur* de G, soit le sous-groupe de complémentation qui fixe G. Puisque le stabilisateur d'un graphe dépend de l'ensemble des arêtes, afin d'en obtenir une bonne description ainsi qu'une bonne description des classes d'équivalence qu'il induit, nous introduisons le concept de *règle de substitution.* Certaines règles de substitution ne dépendent de la relation d'adjacence que localement, sur un sous-ensemble de V, elles sont qualifiées de *locales.* Les règles de substitution locales sont caractérisées dans cette article. La complémentation locale de graphes ayant une 2-coloration de leurs sommets est définie et étudiée d'une manière analogue et nous appellerons leurs orbites de complémentation *orbites de Sabidussi.* Comme application, nous montrons comment le polynôme d'entrelacement de Arratia, Bollobás et Sorkin est un invariant des orbites de Sabidussi. Nous démontrons également de quelle façon les orbites de complémentation sont liées à plusieurs constructions combinatoires de Bouchet telles que les systèmes isotropes, les delta-matroïdes et les multimatroïdes.

Abstract

Local complementation of a simple graph $G = (V, E)$ at a vertex $u \in V$ reverses the adjacency relation of G over the set of neighbors of u. Given a finite set V, local complementation induces a group action on the set of simple graphs defined on a finite vertex set

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†adresse de correspondance : Department of Computer Science and Software Engineering, Concordia University, 1455, de Maisonneuve blvd West, Montréal, Québec, H3G 1M8, Canada

V. Each resulting orbit of graphs is called a *Kotzig orbit*. For a given graph G, the members of its Kotzig orbit can be represented by finite sequences of local complementations, one from each coset of its *complementation stabilizer*, by which we mean the complementation subgroup that fixes G. The complementation stabilizer of a graph depends on its edge set, so in order to obtain a good description of it and of its cosets, we introduce the notion of *substitution rules*. Some rules depend only on the adjacency relation over a subset of V, they are said to be *local*. Local substitution rules are characterized in this paper. The local complementation of graphs with a vertex 2-coloring is defined and explored in an analogous manner and their complementation orbits are called *Sabidussi orbits*. As an application, we note that the vertex-nullity interlace polynomial of Arratia, Bollobás and Sorkin is an invariant of Sabidussi orbits. We also demonstrate how complementation orbits are related to several combinatorial constructions of Bouchet such as isotropic systems, delta-matroids and multimatroids.

1 Introduction

After giving in this section the basic definitions and properties of the local complementation of simple uncolored and bicolored graphs, we introduce Kotzig orbits and Sabidussi orbits in Sections 2 and 3, respectively. Minors of Kotzig orbits and Sabidussi orbits are defined in Section 4 and the relationship between Sabidussi orbits and other combinatorial objects is discussed in Section 5. In Section 6, we make some remarks about Sabidussi orbits and the vertex-nullity interlace polynomial of Arratia *et al.*

1.1 Motivation

The local complementation of simple graphs was introduced by Kotzig in relation to the κ-transformation of the eulerian trails of 4-regular graphs (as mentioned in [8, 15]). It was investigated, amongst others, by Bouchet [7], Fon-der-Flaass [13, 14], Kotzig [17] and Sabidussi [19]. Local complementation is intimately related to the theory of isotropic systems developed by Bouchet [7]. Sabidussi established the basis of local complementation applied to *bicolored graphs* (defined in subsection 1.2) in relation to his work on his Compatibility Conjecture [19]. In that context, the 2-coloring corresponds to the parity of the degrees (even is white, odd is black). The author generalized Sabidussi's local complementation to arbitrary 2-colorings in his thesis [16] and many results appearing in this article are taken from there with the difference that the point of view of group theory adopted here is new and, hopefully, helps to clarify the concepts involved.

Recent work shows a renewed interest in local complementation. For example, Arratia *et al.* [3], motivated by a problem relating to DNA sequencing, rediscovered the Martin polynomial and generalized it to what they call *looped graphs*, which are essentially bicolored graphs where the 2-coloring is encoded by the presence or absence of a loop. In so doing, they defined the interlace polynomial, also called the vertex-nullity polynomial, which is shown to be identical to the Tutte-Martin polynomial of Bouchet in Section 6. In work related to the rank-width of a graph, Oum [18] revisits the work of Bouchet on the i-minors of graphs. In a forthcoming paper, Cada *et al.* [12] extend Bouchet's result on the reduction of prime uncolored graphs [5] to prime bicolored graphs using results presented here.

One of the main results of this paper is the characterization of *local substitution rules* (see definitions 2.4 and 2.5), which allows us to find good sequences of vertices to represent the members of the Kotzig orbit of a graph. In the case of bicolored graphs, we are even able to represent the members of a Sabidussi orbit by subsets of the vertex set, a surprising and esthetically pleasing result. More can be found in the thesis [16], such as a how Sabidussi orbits are related to Sabidussi's Compatibility Conjecture and to the Cycle Double Cover Conjecture, but the results presented here are self-contained and interesting by themselves.

The aims of this paper are to present the fundamental theorems of the theory of local complementation and to give examples of how it is related to other subjects of interest. In particular, we bring attention to the fact that the vertex-nullity polynomial of Arratia *et al.* is an invariant of Sabidussi orbits and that Sabidussi orbits are therefore the natural objects on which to define this polynomial.

1.2 Basic definitions

Any terminology about graphs not defined here is found in Bondy and Murty [4]. Local complementation operations modify graphs but leave their vertex sets unchanged. Therefore, it is natural to consider their action on sets of graphs sharing a common vertex set. In this paper, given a finite set V, a *graph* $G = (V, E)$ has the *vertex* set V and an *edge* set E of unordered pairs of distinct vertices. The edge set E determines a unique irreflexive symmetric relation on V that is called the *adjacency* relation of G. Edges are commonly written as comma-separated vertices inside brackets, i.e. $[u, v]$. A *bicolored graph* $G = (V, E, c)$ is the graph (V, E) with a 2-coloring $c : V \mapsto \{0, 1\}$ of its vertices. In figures, vertices of color 0 are drawn as empty (white) circles and vertices of color 1 are drawn as solid (black) circles (see Figure 2). The colors are referred to as white and black, respectively. When the vertex set V is fixed, as will be the case in our study of local complementation, a graph is completely determined by its edge set and there will be no ambiguity when we refer to a graph $G = (V, E)$ as the graph E. For $A, B \subset V$, $A \times B = \{[u, v] | u \neq v, u \in A, v \in B\}$ and $A^2 = A \times A$ (the complete graph on A). Given $A, B \subset V$ or $A, B \subset E$,

$$A + B = (A \backslash B) \cup (B \backslash A) \text{ (the symmetric difference).}$$

If A or B is a singleton, we omit the braces, i.e. $A + u = A + \{u\}$ and $E + [u, v] = E + \{[u, v]\}$. Recall that the symmetric difference operation is associative and commutative. Some simple observations:

$$B \times A = A \times B = (A \cap B)^2 + (A \backslash B) \times (B \backslash A)$$

$$\text{and } A \times (B + C) = A \times B + A \times C.$$

For a given $G = (V, E)$, the neighborhood of $u \in V$ is $N(u) = N_G(u) = \{v \in V | [u, v] \in E\}$ and the *degree* of u is $d_G(u) = |N_G(u)|$.

Definition 1.1. Given $G = (V, E)$ and $u \in V$, the *local complement at* u of $G = (V, E)$ is the graph

$$Gu = (V, E + N(u)^2).$$

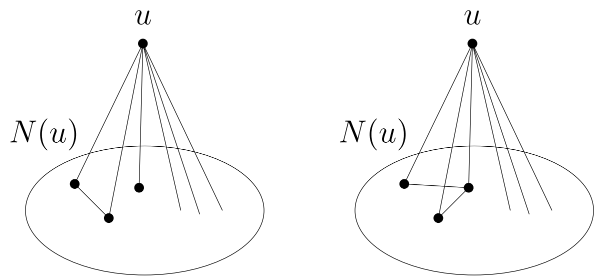

Figure 1: A complementation at u.

Note how the neighborhood of v is changed by a complementation at u:

$$N_{Gu}(v) = N_G(v) + \begin{cases} N_G(u) + v & \text{if } [u, v] \in E \\ \emptyset & \text{otherwise.} \end{cases}$$

From the definition, we see that each vertex determines a local complementation operation. This extends naturally to sequences of vertices, and each sequence will determine a complementation operation.

Definition 1.2. Let $s \in V^+ = \{u_1 u_2 ... u_r | u_i \in V, r \geq 1\}$. The *complement* of $G = (V, E)$ *with respect to* s is the graph

$$Gs = Gu_1 u_2 ... u_r = (...((Gu_1)u_2)...)u_r.$$

Sequences of the form $uvu \in V^+$ play an important role in local complementation and a special notation is reserved for them : we write $[uv]$ for uvu.

For a bicolored graph $G = (V, E, c)$, local complementation operations are not defined in terms of vertices, but instead depend on the choice of an unordered pair of vertices, not necessarily distinct. For ease of notation, when working with bicolored graphs, a complementation with respect to u will mean a complementation with respect to the pair $\{u, u\}$ and a complementation with respect to $[uv]$ will mean a complementation with respect to the pair $\{u, v\}$. Although the notation is similar to the uncolored case, the meaning will be clear from the context and the similarity will prove to be natural and useful.

Definition 1.3. Given a bicolored graph $G = (V, E, c)$ and $u, v \in V$, we denote the *local complement* of G *with respect to* $\{u, v\}$ by $G[uv]$ and when $u = v$ we also use the notation Gu. Let

$$A = N(u)\backslash N(v) + v,\ \ B = N(u) \cap N(v),\ \ C = N(v)\backslash N(u) + u.$$

$G[uv] = G(V, E', c')$ is defined in the following way :

- If $u = v$ is white, then

$$E' = E + N(u)^2 \text{ and } c'(w) = \begin{cases} 1 - c(w) & \text{if } w \in N(u) \\ c(w) & \text{otherwise.} \end{cases}$$

- If $u \neq v$ are black and $[u, v] \in E$, then

$$E' = E + \{u, v\} \times (A \cup C) + A \times B + B \times C + A \times C \text{ and } c' = c.$$

- In all remaining cases,

$$G[uv] = G.$$

The non-trivial local complementations are illustrated in Figure 2 and Figure 3. A dotted line between two sets of vertices indicates that edges between the sets have been "toggled" in the sense that the existing edges were removed and the missing edges were added.

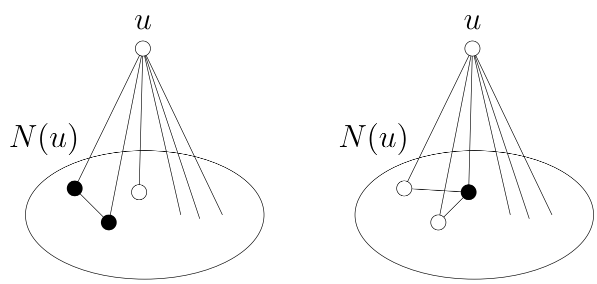

Figure 2: A complementation at white $u = v$.

Definition 1.4. Let $s = p_1 ... p_r$ be a finite sequence of unordered pairs of vertices. The *complement* of $G = (V, E, c)$ with respect to s is the bicolored graph

$$Gs = Gp_1 ... p_r = (...(Gp_1)...)p_r.$$

Definition 1.5. $V(s)$ is the set of vertices appearing in the word s (the *support* of s) as elements of the sequence (uncolored case) or as elements of the pairs (bicolored case).

For example, in the uncolored case, for $s = [uv]w = uvuw$, $V(s) = \{u, v, w\}$ and in the bicolored case, for $s = u[vw] = \{u, u\}\{v, w\}$, $V(s) = \{u, v, w\}$.

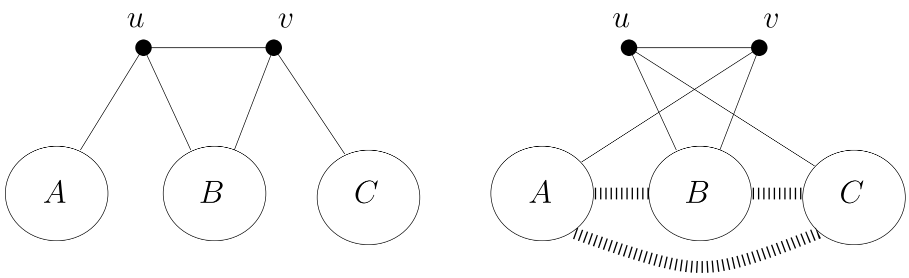

Figure 3: A complementation at adjacent black u, v.

1.3 Fundamental properties

For an uncolored graph $G = (V, E)$, aside from the trivial statements

$$Guu = G \tag{1}$$

$$[u, v] \notin E \Rightarrow Guv = Gvu, \tag{2}$$

it is a known fact that if $[u, v] \in E$ then $Guvu = Gvuv$. This is seen from the symmetry of Lemma 1.6. Therefore we have

$$[u, v] \in E \Rightarrow G[uv] = G[vu] \tag{3}$$

and this result suggests that local complementation operations can be defined for edges. Indeed, this has been done and, up to a relabeling of the vertices u, v, they are called *pivot* or *switching* operations in the literature. We refrain from that here and reserve for bicolored graphs the notion of complementation with respect to pairs of vertices.

The following lemma, which is the usual way to prove (3) (note the symmetry in u and v), establishes the relationship between the local complementation of uncolored and bicolored graphs.

Lemma 1.6. *In a graph $G = (V, E)$, let $[u, v] \in E$, $A = N(u)\backslash N(v)+v$, $B = N(u)\cap N(v)$, $C = N(v)\backslash N(u) + u$, so that $N(u) = A + B + v$ and $N(v) = B + C + u$. Then*

$$Guvu = G + \{u, v\} \times (A \cup C) + A \times B + B \times C + A \times C.$$

Proof.

$$Gu + G = (A + B + v)^2 = A^2 + B^2 + A \times B + (A + B) \times v \tag{4}$$

$$\begin{aligned} Guv + Gu &= N_{Gu}(v)^2 = (A + C + u)^2 \\ &= A^2 + C^2 + A \times C + (A + C) \times u \end{aligned} \tag{5}$$

$$\begin{aligned} Guvu + Guv &= N_{Guv}(u)^2 = (B + C + v)^2 \\ &= B^2 + C^2 + B \times C + (B + C) \times v \end{aligned} \tag{6}$$

Adding (4)+(5)+(6), we get

$$Guvu + G = A \times B + B \times C + A \times C + (A + C) \times u + (A + C) \times v.$$

□

This lemma explains why a complementation at a pair of adjacent black vertices $\{u, v\}$ can be interpreted as the complementation of the underlying uncolored graph with respect to the sequence uvu or vuv. With this in mind, (1), (2) and (3) also hold in the context of bicolored graphs.

Another fundamental property of the local complementation of uncolored graphs is the following.

$$\text{(uncolored) } [u, v], [u, w] \in E, v \neq w \Rightarrow G[uv][wv] = G[wu] \tag{7}$$

This was shown by Arratia *et al.* in their work on the interlace polynomial [2] and a nice algebraic proof of this result is given by Oum [18]. The author proved independently the following equivalent statement in [16].

$$\text{(uncolored) } [u, v], [v, w], [u, w] \in E \Rightarrow G[uv][vw][wu] = G \tag{8}$$

Equivalence follows from the fact that if $[v, w] \notin E$ then (8) can be applied in Gu so that

$$G = Guu = (Gu)[uv][vw][wu]u = Guuvuvwvwuwu = G[vu][wv][uw].$$

Another useful equivalent formulation of (7) is the following.

$$\text{(uncolored) } [u, v], [v, w] \in E, u \neq w \Rightarrow Guvw = G[vw]u \tag{9}$$

In the case of bicolored graphs, we have the following corresponding properties.

$$u, v, w \text{ black}, [u, v], [u, w], \in E, v \neq w \Rightarrow G[uv][vw][wu] = G \tag{10}$$

$$u \text{ white}, v, w \text{ black}, [u, v], [v, w] \in E, u \neq w \Rightarrow Guvw = G[vw]u \tag{11}$$

Proof of (10) *and* (11). From (7) and (9), the underlying uncolored graphs are equal and there remains only to check that the colorings are equal in (11). Let $C(s)$ be the set of vertices with different colors in G and Gs. We have that $C(u) = N(u)$, $N_{Gu}(v) = N(v) + N(u) + v$, $C(uv) = C(u) + N_{Gu}(v) = N(v) + v$.

If $[u, w] \in E$,

$$N_{Guv}(w) = N_{Gu}(w) = N(w) + N(u) + w,$$

$$C(uvw) = C(uv) + N_{Guv}(w) = N(u) + N(v) + N(w) + v + w,$$

$$C([vw]u) = N_{G[vw]}(u) = N(u) + N(v)\backslash N(w) + w + N(w)\backslash N(v) + v$$

$$= N(u) + N(v) + N(w) + v + w.$$

If $[u, w] \notin E$,

$$N_{Guv}(w) = N_G(w) + N_{Gu}(v) + w = N(u) + N(v) + N(w) + v + w,$$

$$C(uvw) = C(uv) + N_{Guv}(w) = N(u) + N(w) + w,$$

$$C([vw]u) = N_{G[vw]}(u) = N(u) + N(w) + w.$$

In all cases $C(uvw) = C([vw]u)$. □

2 Kotzig orbits

In this section, all graphs considered are uncolored.

2.1 Kotzig orbits and graph stabilizers

Consider the set of words $V^* = V^+ \cup \{\epsilon\}$ of all finite sequences of elements of V including the empty word ϵ. Letting $uu = \epsilon$ for each $u \in V$, the set V^* can be considered as a group with concatenation as the group operation: the unit of the group is ϵ and the inverse s^{-1} of a word $s = u_1 ... u_r$ is $s^{-1} = u_r ... u_1$. Let $\mathcal{G}_V$ be the set of all graphs with vertex set V. By the definition of local complementation and because of (1), it is natural to view complementation with respect to words as the group action of V^* on $\mathcal{G}_V$.

Definition 2.1. The *Kotzig orbit* of a graph $G = (V, E)$ is the set

$$GV^* = \{Gs | s \in V^*\} \subset \mathcal{G}_V.$$

An important problem in the subject of local complementation is to determine when two graphs of $\mathcal{G}_V$ can be obtained from one another by a sequence of local complementations or, using our terminology, when two graphs belong to the same Kotzig orbit. This can be done in polynomial time, as was proved by Bouchet [10]. A generalized proof for directed graphs is given by Fon-der-Flaass [13].

Another important problem is to find good elements of V^* to represent the members of the Kotzig orbit of a given graph. Obviously, we would favor short words over arbitrarily long ones and we would like these words to be "efficient" in some sense. Intuitively, we would like to avoid having any vertex appearing more times than necessary. This is the motivation for the study of the *stabilizer* of a graph and the introduction of *substitution rules* in the next subsection.

Definition 2.2. The *stabilizer* of $G = (V, E)$ is the subgroup of V^*

$$\Gamma_G = \{s \in V^* | Gs = G\}.$$

An important point is that, except in the most trivial cases, Γ_G is not a normal subgroup of V^* because it depends on the adjacency relation of G. Therefore the cosets V^*/Γ_G do not form a group.

Definition 2.3. Given the graph $G = (V, E)$, we define an equivalence relation $\sim_G$ on V^* by $s \sim_G s'$ if and only if $Gs = Gs'$.

With this definition, $\Gamma_G = \{s \in V^* | s \sim_G \epsilon\}$.

2.2 Substitution rules

It is obvious that given $u, v, w \in V$, we can deduce from (1) that the words uu, $uvvu$, $uvwuuvvwvu$ are all in Γ_G. Just this one property ensures that Γ_G contains an infinity of words and we have not yet taken into account properties (2), (3), (7). To get a good description of the cosets of Γ_G, we need to describe Γ_G itself and this is made possible by the concept of substitution rules. The main result of this section is Theorem 2.13, which says that to know the cosets of Γ_G and to obtain a good description of the Kotzig orbit, it suffices to look at the words of V^* that are *reduced* with respect to G (Definition 2.10) and those reduced words are nicely related to one another by Proposition 2.15.

Definition 2.4. Let $G = (V, E)$ be a graph variable and let s be a word on a set of vertex variables over V. Let P be a logical formula dependent on G (a property of G). The couple $R : (P, s)$ is a *substitution rule* if

$$P(G) \Rightarrow s \sim_G \epsilon.$$

Such a rule will often be written $R : P \Rightarrow s \sim_G \epsilon$. Unless otherwise specified, we assume rules to be non-trivial, i.e. at least one graph G satisfies P.

Given a substitution rule $R : P \Rightarrow s \sim_G \epsilon$, some fixed graph G and words $s_1 = s'ss''$ and $s_2 = s's''$ such that $P(Gs')$ is true, we deduce that $s_1 \sim_G s_2$. To emphasize that R was used, we write $s_1 \overset{R}{\sim}_G s_2$. The following substitution rules are easily deduced from (1), (2), (3) and (7).

$$uu \sim_G \epsilon, \tag{R1}$$
$$u \neq v \text{ and } [u, v] \notin E(G) \Rightarrow uvuv \sim_G \epsilon, \tag{R2}$$
$$[u, v] \in E \Rightarrow [uv][vu] \sim_G \epsilon, \tag{R3}$$
$$v \neq w, [u, v], [u, w] \in E \Rightarrow [uv][wv][wu] \sim_G \epsilon, \tag{R4}$$

The following two rules are fundamentally different from the previous ones, can you see in what way?

$$d(u) \leq 1 \Rightarrow u \sim_G \epsilon, \tag{R5}$$
$$N(u) = N(v) \Rightarrow uv \sim_G \epsilon. \tag{R6}$$

In the first rules, the adjacency relation is considered only on a subset of vertices. In the last two, say R5 for example, to ensure that $d(u) \leq 1$, we have to look at the relationship of u with every vertex of the graph.

Definition 2.5. A substitution rule $P \Rightarrow s \sim_G \epsilon$ is *local* if for any two graphs G and H such that G is an induced subgraph of H,

$$P(G) \Rightarrow s \sim_H \epsilon.$$

A non-local rule is *global*.

Thus R1 to R4 are local rules and R5 and R6 are global rules.

Definition 2.6. Given a word $s \in V^*$ and a set of substitution rules $\mathcal{R}$, $\mathcal{R}_G(s)$ is the set of words which can be deduced to be equivalent to s using the rules in $\mathcal{R}$, i.e., $s' \in \mathcal{R}_G(s) \iff \exists R_1, ..., R_n \in \mathcal{R}$ and $s_0, ..., s_n \in V^*$, $n \geq 0$, such that $s_0 = s$, $s_n = s'$, and $s_i \overset{R_{i+1}}{\sim}_G s_{i+1}$, $i = 0, ..., n-1$.

Definition 2.7. $Loc_G(s) = \{R1, R2, R3, R4\}_G(s)$. We write $s \overset{loc}{\sim}_G t$ if $s \in Loc_G(t)$.

Definition 2.8. Let $\mathcal{R}$ be a set of substitution rules. The set of substitution rules *generated* by $\mathcal{R}$, written $\langle \mathcal{R} \rangle$, is the set of rules (P, s) such that for every graph G satisfying P, we have $s \in \mathcal{R}_G(\epsilon)$. $\mathcal{R}$ is *independent* if no proper subset of $\mathcal{R}$ generates $\langle \mathcal{R} \rangle$.

From the definition of local substitution rules follows that:

Proposition 2.9. *Local rules generate local rules.*

Definition 2.10. A word $s \in V^*$ is *reduced* with respect to $G = (V, E)$ if it can be written $s = s_1 ... s_r$, where no letter appears in different s_i's and each s_i either consists of a single vertex or can be expressed as $s_i = [uv]$, where $[u, v] \in E(Gs_1 ... s_{i-1})$.

Proposition 2.11. *A substitution rule $P \Rightarrow s \sim_G \epsilon$ where s is a non-empty reduced word is global.*

Proof. Let $G = (V, E)$ be a graph satisfying P. Let $u, v \notin V$. If s ends with a single occurrence letter w, let $E' = E \cup \{[u, w], [v, w]\}$. If not, we have $s = s'wxw$ with $w, x \notin V(s')$, in which case let $E' = E \cup \{[u, w], [v, x]\}$. By construction, G is an induced subgraph of $H = (V \cup \{u, v\}, E')$ but $[u, v] \notin H$ while $[u, v] \in Hs$. Thus $Hs \neq H$. ☐

Lemma 2.12. *Given $G = (V, E)$ and $u_0, v, w \in V$ subject to the condition that u_0, v, w be distinct for* (12e) *to* (12g)*, the following hold.*

$$u_0 u_0 \overset{loc}{\sim}_G \epsilon \tag{12a}$$

$$[v, u_0] \notin E \Rightarrow u u_0 \overset{loc}{\sim}_G u_0 u \tag{12b}$$

$$v v u_0 \overset{loc}{\sim}_G u_0 \tag{12c}$$

$$[v, u_0] \in E \Rightarrow u_0 v u_0 \overset{loc}{\sim}_G v u_0 v \tag{12d}$$

$$[v, w], [v, u_0] \notin E \Rightarrow v w u_0 \overset{loc}{\sim}_G w u_0 v \tag{12e}$$

$$[w, u_0], [v, w] \in E \Rightarrow v w u_0 \overset{loc}{\sim}_G w u_0 w v \tag{12f}$$

$$[v, u_0] \in E, [v, w] \in E(G u_0) \Rightarrow v w u_0 \overset{loc}{\sim}_G u_0 v w v \tag{12g}$$

Proof. We prove (12e) to (12g), the other statements are straightforward.
(12e) Since $[v, u_0] \notin E(Gw)$, we have, by (12c), that $vw \overset{loc}{\sim}_G wv$ and $vu_0 \overset{loc}{\sim}_{Gw} u_0 v$, so that $vwu_0 \overset{loc}{\sim}_G wu_0 v$.
(12f) $vwu_0 \overset{R4}{\sim}_G [wu_0][vu_0][vw]vwu_0 \overset{R1}{\sim}_G wu_0 wv$.
(12g) $vwu_0 \overset{R1}{\sim}_G u_0 u_0 vwu_0$ and, since $[v, w], [u_0, v] \in E(Gu_0)$, we have, by (12f), that $u_0 vw \overset{loc}{\sim}_{Gu_0} vwvu_0$ and thus $vwu_0 \overset{loc}{\sim}_G u_0(vwvu_0)u_0 \overset{loc}{\sim}_G u_0 vwv$. ☐

Theorem 2.13. *Given $G = (V, E)$, $s \in V^*$ and $u \in V$, there exists a reduced $s' \overset{loc}{\sim}_G s$ such that $V(s') \subset V(s)$, and u appears in position 1 or 2 of s', if at all.*

Proof. By way of contradiction, suppose that G, u and s constitute a counter-example with $\lambda(s) := |V(s)|$ minimal. Clearly s is non-empty. Choose $u_0 \in V(s)$, with the restriction that $u_0 = u$ if $u \in V(s)$. Writing $\rho(v, t)$ for the position of the last occurrence of v in t, we can suppose without loss of generality that $r := \rho(u_0, s) \leq \rho(u_0, s')$ for all $s' \overset{loc}{\sim}_G s$ such that $V(s') \subset V(s)$. Suppose that $r > 2$ and consider the subwords of s of length 2 and 3 ending with the last occurrence of u_0. For all possible edge sets just prior to complementation to the subword considered, Lemma 2.12 ensures that we can find a word $s' \overset{loc}{\sim}_G$ with $V(s') = V(s)$ and $\rho(u_0, s') < r$, a contradiction. Therefore we must have $r \leq 2$. By the minimality of $\lambda(s)$, s cannot have the prefix u_0u_0 (if $s = u_0u_0s''$ then $s \sim_G s''$ with $\lambda(s'') < \lambda(s)$). If $s = u_0s''$, then by the minimality of $\lambda(s)$ we know that s'' can be replaced by a word reduced with respect to Gu_0 and not containing u_0, resulting in a reduced word equivalent to s, a contradiction. Thus s is of the form $s = vu_0s''$ with $[v, u_0] \in E(G)$ and $u_0 \notin V(s'')$. Now consider the graph $H = Gvu_0$. We can find a reduced word $t \overset{loc}{\sim}_H s''$ with $V(t) \subset V(s'')$ and $v \notin V(t)$ or v in position 1 or 2. However, v cannot be absent from t or else $s \overset{loc}{\sim}_G vu_0t$, a reduced word. If $t = vt'$ then $s \overset{loc}{\sim}_G [vu_0]t'$, a reduced word. The only remaining possibility is $t = wvt'$ with $[v, w] \in E(H)$ (if $[v, w] \notin E(H)$, then $t \overset{loc}{\sim}_H vwt'$ for which we can apply the preceding argument). Knowing that $[v, u_0] \in E(G)$ and $[v, w] \in E(Gvu_0)$, we must have $[w, u_0] \in E(G)$. From (12f), we have that $s \overset{loc}{\sim}_G vu_0wvt' \overset{loc}{\sim}_G (u_0wu_0v)vt' \overset{loc}{\sim}_G [wu_0]t'$. If $w \notin V(t')$, then $[wu_0]t'$ is reduced with respect to G. If $w \in V(t')$ then, since t is reduced with respect to H, t is of the form $t = wvwt''$ and $s \overset{loc}{\sim}_G wu_0t''$, which is reduced. Therefore no counter-example exists. $\square$

Proposition 2.14. *The rules*

$$uu \sim_G \epsilon, \tag{R1}$$

$$u \neq v \text{ and } [u, v] \notin E(G) \Rightarrow uvuv \sim_G \epsilon, \tag{R2}$$

$$[u, v] \in E \Rightarrow [uv][vu] \sim_G \epsilon, \tag{R3}$$

$$v \neq w, [u, v], [u, w] \in E \Rightarrow [uv][wv][wu] \sim_G \epsilon, \tag{R4}$$

form an independent generating set of the local rules.

Proof. We first show independence. Consider $G = (\{u, v\}, \{[u, v]\})$. Since $GV^* = \{G\}$, $\{\text{R1,R2,R4}\}_G(\epsilon) = \{\text{R1}\}_G(\epsilon)$ and any word in $\{\text{R1}\}_G(\epsilon)$ will contain an even number of occurrences of the letter u. Thus any independent generating subset of the four rules must contain R3.

Now let $G = (\{u, v, w\}, \{[u, v], [u, w], [v, w]\})$. Any word in $\{\text{R1,R2,R3}\}_G(\epsilon)$ has an even number of letters, so that R4 is also essential. If we let $G = (\{u\}, \emptyset)$, then $\{\text{R2,R3,R4}\}_G(\epsilon) = \{\epsilon\}$, thus R1 is essential. Finally, let $G = (\{u, v\}, \emptyset)$. Defining the total order $u < v$ on $V(G)$, let the sign of a word $s = u_1u_2...u_n$ in V^* be $\sigma(s) = (-1)^\alpha$

where $\alpha = \text{card}\{(i,j)|u_i < u_j, 1 \leq i < j \leq n\}$. By induction on the length of words, we can show that for any $s \in \{\text{R1,R3,R4}\}_G(\epsilon) = \{\text{R1}\}_G(\epsilon)$, we have $\sigma(s) = 1$. Since $\sigma(uvuv) = -1$, this completes the proof of independence.

We know from Proposition 2.9 that $\langle\{\text{R1,R2,R3,R4}\}\rangle$ is a set of local rules. Let $P \Rightarrow s \sim_G \epsilon$ be a local rule. Consider $V = V(s)$ and $\{G_i\}_{i\in I}$ the family of graphs on the vertex set V satisfying P. By Theorem 2.13, for any rule $\mathcal{R}_i : G|_V = G_i \Rightarrow s \sim_G \epsilon$ there exists a rule $\mathcal{R}'_i : G|_V = G_i \Rightarrow s' \sim_G \epsilon$ in $\langle\{\mathcal{R}_i,\text{R1,R2,R3,R4}\}\rangle$ where s' is reduced. Since $\mathcal{R}'_i$ is local, Proposition 2.11 forces $s' = \epsilon$. Thus $\mathcal{R}'_i \in \langle\{\text{R1,R2,R3,R4}\}\rangle$, and since every substitution in the proof of Theorem 2.13 is reversible, we have $\mathcal{R}_i \in \langle\{\text{R1,R2,R3,R4}\}\rangle$. Since $P \Rightarrow s \sim_G \epsilon$ is generated by the $\mathcal{R}_i$'s, we conclude that it is in $\langle\{\text{R1,R2,R3,R4}\}\rangle$. ☐

Proposition 2.15. *If s, s' are reduced words such that $s' \overset{loc}{\sim}_G s$, then $V(s) = V(s')$.*

Proof. Suppose, by way of contradiction, that there is a $u \in V(s)\backslash V(s')$. Then $s's^{-1} \sim_G \epsilon$ and a reduced word $t \overset{loc}{\sim}_G s's^{-1}$ given by Theorem 2.13 will contain u. But this would mean that local rules generate a global rule of the form $P \Rightarrow t \sim_G \epsilon$, contradicting Proposition 2.9. ☐

3 Sabidussi orbits

In this section, all graphs are bicolored.

3.1 Definitions

Let V_p be the set of unordered pairs of vertices of V with repetitions allowed. Then, similar to the uncolored case, local complementation can be seen as the group action of V_p^* on $\mathcal{B}_V$, the set of all bicolored graphs with vertex set V.

Definition 3.1. The *Sabidussi orbit* of a bicolored graph $G = (V, E, c)$ is the set

$$GV_p^* = \{Gs | s \in V_p^*\} \subset \mathcal{B}_V.$$

Definition 3.2. The *stabilizer* of $G = (V, E, c)$ is the subgroup of V_p^*

$$\Gamma_G = \{s \in V_p^* | Gs = G\}.$$

Definition 3.3. Given $G = (V, E, c)$, we define an equivalence relation $\sim_G$ on V_p^* by $s \sim_G s'$ if and only if $Gs = Gs'$.

With this definition, $\Gamma_G = \{s \in V_p^* | s \sim_G \epsilon\}$.

3.2 Complementation sets

We want to obtain for Sabidussi orbits the concept equivalent to the reduced words that we found for Kotzig orbits. In the discussion following Lemma 1.6, we point out the connection between the local complementation of uncolored graphs and the local complementation of bicolored graphs, namely that to each local complementation of a bicolored graph there corresponds, for the underlying uncolored graph, a complementation with respect to a word of the form ϵ, u or uvu. With this correspondence in mind, the concepts and proofs in this section, although technically different from those of the preceding section, follow the same ideas.

Definition 3.4. A word $s \in V_p^*$ is *reduced* with respect to G if $s = p_1 ... p_r$, where no vertex appears in different p_i's and each p_i is of the form $p_i = u$ where u is white in $Gp_1 ... p_{i-1}$ or $p_i = [uv]$, where u, v are adjacent black vertices in $Gp_1 ... p_{i-1}$.

Substitution rules and local rules are defined just as in the uncolored case. The following are local substitution rules. Rules T1 and T2 reflect the fact that, by definition, some local complementations do not change the graph.

$$u \text{ black} \Rightarrow u \sim_G \epsilon \tag{T1}$$

$$u \neq v \text{ not adjacent black vertices} \Rightarrow [uv] \sim_G \epsilon \tag{T2}$$

$$[uv][uv] \sim_G \epsilon, \tag{C1}$$

$$[u, w], [u, x], [v, w], [v, x] \notin E(G) \Rightarrow [uv][wx][uv][wx] \sim_G \epsilon, \tag{C2}$$

$$u, v, w \text{ black}, [u, v], [u, w], \in E, v \neq w \Rightarrow [uv][vw][uw] \sim_G \epsilon \tag{C3}$$

$$u \text{ white}, v, w \text{ black}, [u, v], [v, w] \in E, u \neq w \Rightarrow uvwu[vw] \sim_G \epsilon \tag{C4}$$

Definition 3.5. $Loc_G(s) = \{T1, T2, C1, C2, C3, C4\}_G(s)$. We write $s \overset{loc}{\sim}_G t$ if $s \in Loc_G(t)$.

Lemma 3.6. *Let $G = (V, E, c)$ and consider $p_1 p_2 \in V_p^*$ such that, in the appropriate graph (G when considering p_1, Gp_1 when considering p_2), p_i is either of the form $p_i = u = [uu]$ where u is white or $p_i = [uv]$ where u, v are adjacent black vertices. Subject to the condition* *that* u_0, u, v, w *be distinct in* (13c) *to* (13i)*, the following hold.*

$$[uu_0][uu_0] \overset{loc}{\sim}_G \epsilon \text{ (includes the case } u_0u_0 \overset{loc}{\sim}_G \epsilon\text{)} \tag{13a}$$

$$[u,w],[u,u_0],[v,w],[v,u_0] \notin E \Rightarrow [uv][wu_0] \overset{loc}{\sim}_G [wu_0][uv] \tag{13b}$$

$$[u_0,v],[v,w] \in E \Rightarrow [vw]u_0 \overset{loc}{\sim}_G u_0vw \tag{13c}$$

$$[u,u_0] \in E \text{ and either } [u,v] \text{ or } [vu_0] \in E \Rightarrow u[vu_0] \overset{loc}{\sim}_G u_0vu \tag{13d}$$

$$[u,u_0] \notin E, [u,v] \in E \Rightarrow u[vu_0] \overset{loc}{\sim}_G vu_0u \tag{13e}$$

$$[vu_0][wu_0] \overset{loc}{\sim}_G [vw] \tag{13f}$$

$$[vw][vu_0] \overset{loc}{\sim}_G [wu_0] \tag{13g}$$

$$[u,u_0] \in E \Rightarrow [uv][wu_0] \overset{loc}{\sim}_G [uu_0][vw] \tag{13h}$$

$$[u,u_0],[v,u_0] \notin E, [u,w] \in E \Rightarrow [uv][wu_0] \overset{loc}{\sim}_G [wu_0][uv] \tag{13i}$$

Proof. We prove (13d) to (13i).
(13d) $u[vu_0] \overset{C4}{\sim}_G u(uu_0vu[vu_0])[vu_0] \overset{C1}{\sim}_G u_0vu$.
(13e) $u[vu_0] \overset{C4}{\sim}_G u(uvu_0u[vu_0])[vu_0] \overset{C1}{\sim}_G vu_0u$.
(13f) $[vu_0][wu_0] \overset{C3}{\sim}_G ([vw][wu_0][vu_0])[vu_0][wu_0] \overset{C1}{\sim}_G [vw]$.
(13g) By (13f), $[vw][vu_0] \overset{loc}{\sim}_G [wu_0]$.
(13h) $[uv][wu_0] \overset{C3}{\sim}_G [uu_0][vu_0][uv][uv][wu_0] \overset{C1}{\sim}_G [uu_0][vu_0][wu_0]$
$\overset{C4}{\sim}_G [uu_0]([vw][wu_0][vu_0])[vu_0][wu_0] \overset{C1}{\sim}_G [uu_0][vw]$.
(13i) By (13h), $[uv][wu_0] \overset{loc}{\sim}_G [uw][vu_0] \overset{loc}{\sim}_G [wu_0][uv]$. ☐

Theorem 3.7. *Given* $G = (V, E, c)$*,* $s \in V_p^*$ *and* $u \in V$*, there exists a reduced* $s' \overset{loc}{\sim}_G s$ *such that* $V(s') \subset V(s)$ *and such that, if* $u \in V(s')$*, then* s' *is of the form* us''*,* vus'' *or* $[vu]s''$*.*

Proof. By way of contradiction, suppose that G, u, and $s = p_1...p_k$ constitute a counterexample with $\lambda(s) := |V(s)|$ minimal. Clearly s is non-empty. Choose $u_0 \in V(s)$, with the restriction that $u_0 = u$ if $u \in V(s)$. Writing $\rho(v,t)$ for the index of the last pair of t containing v, we can suppose without loss of generality that $r := \rho(u_0, s) \le \rho(u_0, s')$ for all $s' \overset{loc}{\sim}_G s$ such that $V(s') \subset V(s)$. By $T1$ and $T2$, this ensures that, for $1 \le i \le r$, if p_i is of the form $p_i = u$ then u is white in $Gu_1...u_{i-1}$ and if p_i is of the form $p_i = [vw]$ with $v \ne w$ then v and w are adjacent black vertices in $Gu_1...u_{i-1}$. We can also choose s so that if there exists $s' = p'_1...p'_l \overset{loc}{\sim}_G s$ satisfying $V(s') = V(s)$, $\rho(u_0, s') = r$ and such that $p'_r = u_0$ then $p_r = u_0$. Suppose that $\rho(u_0, s) \ge 2$. Consider the subwords of s of length 2 ending with the last pair containing u_0. For all possible edge sets just prior to complementation to the subword considered, Lemma 3.6 ensures that we can find a word $s' \overset{loc}{\sim}_G$ with $V(s') = V(s)$ and $\rho(u_0, s') < r$, a contradiction. Therefore, s must be of the form $s = u_0s'$, $s = vu_0s'$ or $s = [vu_0]s'$. In each case, by the minimality of $\lambda(s)$, we can suppose that s' is reduced (with respect to Gu_0, Gvu_0 and $G[vu_0]$, respectively) with s' of the form $s' = vt$ or $s' = wvt$ or $s' = [wv]t$ or $v \notin V(s')$. If $s = u_0 s'$, then s is also reduced, a contradiction. If $v \notin V(s')$, we obtain the same contradiction. If $s = vu_0 s'$ with $[vu_0] \notin E$, then, by (13b), $s \overset{loc}{\sim}_G u_0 v s'$, a prior case. Therefore, we have $[v, u_0] \in E$. If v is white in G, using Lemma 3.6, $s = vu_0[wv]t \overset{loc}{\sim}_G v(vwu_0)t \overset{C1}{\sim}_G wu_0 t$, a contradiction. If v is black in G, $s = [vu_0][wv]t \overset{loc}{\sim}_G [wu_0]t$, a contradiction. Therefore no counter-example exists. □

Theorem 3.8. *If s and t are reduced with respect to a bicolored graph G, then $s \overset{loc}{\sim}_G t$.*

Proof. Use induction on $\lambda(s) := |V(s)|$. If $s = us'$, then u is white in G, and applying Theorem 3.7 to t, $t \overset{loc}{\sim}_G ut'$. If s is of the form $[uv]s'$ then u is black in G, and applying Theorem 3.7 to t, $t \overset{loc}{\sim}_G [uw]t'$. If $w \neq v$, apply Theorem 3.7 again to get $t \overset{loc}{\sim}_G [uw][vx]t''$ and finally, from (13h) in Lemma 3.6, $t \sim_G [uv][wx]t''$. By changing the reference graph to Gu or $G[uv]$ accordingly, the problem reduces to words for which λ is smaller. □

Given a bicolored graph G, we are now justified to speak about complementation with respect to subsets of V.

Definition 3.9. A set $S \subset V$ is a *complementation set* of a bicolored graph $G = (V, E, c)$ if there exists a reduced word $s \in V_p^*$ such that $V(s) = S$. In that case, the *complement of G with respect to S* is $GS := Gs$.

The proofs of the two following propositions are similar to the uncolored case and are omitted.

Proposition 3.10. *The rules*

$$u \text{ black} \Rightarrow u \sim_G \epsilon \tag{T1}$$
$$u \neq v \text{ not adjacent black vertices} \Rightarrow [uv] \sim_G \epsilon \tag{T2}$$
$$[uv][uv] \sim_G \epsilon, \tag{C1}$$
$$[u, w], [u, x], [v, w], [v, x] \notin E(G) \Rightarrow [uv][wx][uv][wx] \sim_G \epsilon, \tag{C2}$$
$$u, v, w \text{ black}, [u, v], [u, w], \in E, v \neq w \Rightarrow [uv][vw][uw] \sim_G \epsilon \tag{C3}$$
$$u \text{ white}, v, w \text{ black}, [u, v], [v, w] \in E, u \neq w \Rightarrow uvwu[vw] \sim_G \epsilon \tag{C4}$$

form an independent generating set of the local rules.

Proposition 3.11. *If s, s' are reduced words such that $s' \overset{loc}{\sim}_G s$, then $V(s) = V(s')$.*

Proposition 3.12. *Let S, S' be complementation sets of $G = (V, E, c)$ and let $H = GS$, then*

$$GS' = H(S + S').$$

Proof. Let $s \in V_p^*$ be reduced with respect to H and let $s' \in V_p^*$ be reduced with respect to G with $Hs = G$ and $Gs' = GS'$. We have $GS' = Hss'$ and by Theorem 3.7 there exists a word $t \overset{loc}{\sim}_H ss'$ reduced with respect to H. We first show that if $u \in V(s) \cap V(s')$ then $u \notin V(t)$. By Theorem 3.7 applied to s' and s^{-1} with respect to G, we can suppose that if u is white in G then $s = s_1 u$ and $s' = us_2$ and if u is black in G then $s = s_1[uv]$ and $s' = [uw]s_2$ (with u, v, w black in G). Therefore we have either $ss' \overset{C1}{\sim}_H s_1 s_2$ or, by (13f), $ss' \overset{loc}{\sim}_H s_1[vw]s_2$ so there is a reduced $t' \overset{loc}{\sim}_H t$ with $u \notin V(t')$. By Proposition 3.11, $u \notin V(t)$. Now we let $u \in V(s) + V(s')$ and we show that $u \in V(t)$. Without loss of generality $u \in V(s)$. If u is white in H, we can suppose that $s = us_1$ and by Theorem 3.7, there is $t' \overset{loc}{\sim}_{Hu} s_1 s'$ that is reduced with respect to Hu so that ut' is reduced with respect to H and by Proposition 3.11, $u \in V(t)$. If u is black in H, we can suppose that $s = [uv]s_1$ and then find some $t' \overset{loc}{\sim}_{H[uv]} s_1 s'$ reduced with respect to $H[uv]$. If $v \notin V(t')$, $[uv]t'$ is reduced with respect to H and $u \in V(t)$. If $v \in V(t')$ we can suppose that $t' = [vw]t''$ and thus, by (13f), $s = [uv][vw]t'' \overset{loc}{\sim}_H [uw]t''$ is reduced with respect to H and $u \in V(t)$. □

4 Minors

Bouchet introduced the i-minor of a graph in [5] as an analog to the concept of the minor of a matroid. As his definition involves local complementation, i-minors are implicitly defined for Kotzig orbits. In this section, we define explicitly the minors of Kotzig and Sabidussi orbits and give some elementary properties of minors.

Given a complementation orbit $\mathcal{O}$ (either a Kotzig orbit or a Sabidussi orbit) and $V' \subset V$, it is easy to see that the set $\{G - V' | G \in \mathcal{O}\}$ is stable under the action of complementation and is therefore a union of orbits.

Definition 4.1. Given a Kotzig orbit (respectively, Sabidussi orbit) $\mathcal{O}$ and $S \subset V$, the Kotzig orbits (respectively, Sabidussi orbits) that partition $\{G - S | G \in \mathcal{O}\}$ are called the *S-minors* of $\mathcal{O}$. A $\{u\}$-minor (or u-minor) is also called an *elementary minor* of $\mathcal{O}$ at u.

4.1 Minors of Kotzig orbits

Bouchet proved the following using the properties of isotropic systems [7]. We give a direct proof.

Theorem 4.2. *For any $u \in V$, a Kotzig orbit $\mathcal{O}$ has at most three u-minors. Furthermore, given $G = (V, E) \in \mathcal{O}$, they are*

$$\mathcal{O}_1 = \{Gs - u | u \notin V(s), s \in V^*\}, \ \mathcal{O}_2 = \{Gus - u | u \notin V(s), s \in V^*\}$$

and, if not empty,

$$\mathcal{O}_3 = \{Gvus - u | [u, v] \in E, u \notin V(s), s \in V^*\}.$$

Proof. First note that, by Theorem 2.13, $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = \{G - u | G \in \mathcal{O}\}$. For $u \notin V(s)$, $Gs - u = (G - u)s$, so complementation acts transitively on $\mathcal{O}_1$ and $\mathcal{O}_2$. There remains to show that complementation acts transitively on $\mathcal{O}_3$. Let $[u, v], [u, w] \in E$ and consider $vus, wus' \in V^*$ with $u \notin V(s) \cup V(s')$. Let $H = Gvus$ so that $Gwus' = Hs^{-1}uvwus'$. If $v = w$ then $Gwus' = Hs^{-1}s'$. Otherwise we must have $[u, v], [v, w] \in E(Hs^{-1})$ and, by (12f), $Gwus' = Hs^{-1}[vw]uus' = Hs^{-1}[vw]s'$. Therefore $Gwus' - u$ is in the same Kotzig orbit as $H - u$. ☐

4.2 Minors of Sabidussi orbits

Theorem 4.3. *For any $u \in V$, a Sabidussi orbit $\mathcal{O}$ has at most two u-minors. Furthermore, given $G = (V, E) \in \mathcal{O}$, they are*

$$\mathcal{O}_1 = \{GS - u | u \notin S \textit{ and } S \textit{ is a complementation set of } G\}$$

and, if not empty,

$$\mathcal{O}_2 = \{GS - u | u \in S \textit{ and } S \textit{ is a complementation set of } G\}.$$

Proof. Given complementation sets S, S' of G such that GS, GS' belong to the same $\mathcal{O}_i$, we have, by Proposition 3.12, that $GS' = (GS)(S + S')$ with $u \notin S + S'$. Therefore $GS' - u = (GS - u)(S + S')$ and complementation acts transitively on $\mathcal{O}_1$ and $\mathcal{O}_2$. Since $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, this completes the proof. ☐

Corollary 4.4. *Given $S \subset V$ with $|S| = k$, the number of S − minors of a Sabidussi orbit $\mathcal{O}$ is bounded by 2^k.*

Proof. The proof is a simple induction on k. ☐

5 Complementation orbits and isotropic systems

This section presents the relationship between complementation orbits and several combinatorial structures introduced by Bouchet. Isotropic systems (Definition 5.1) are a generalization of 4-regular graphs and binary matroids (see Bouchet [7]). Matroids and isotropic systems are in turn generalized in [11] by multimatroids (Definition 5.7). Bouchet showed in [7] that isotropic systems are essentially the same objects as Kotzig orbits. We show that Sabidussi orbits can be considered as a subset of the 2-matroids (Definition 5.8). From this relationship, we deduce in the next section that the interlace polynomial of Arratia *et al.* is identical to the Tutte-Martin polynomial of Bouchet.

5.1 Isotropic systems

Let $K = \{0, \alpha, \beta, \gamma\} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and define a product $K \times K \mapsto GF(2)$ by $xy = 1$ if and only if $0 \neq x \neq y \neq 0$. Given a set V of cardinality n, say, for simplicity, $V = \{1, ..., n\}$, we consider the vector space K^V of dimension $2n$ over the field $GF(2)$ together with the bilinear form $\langle A, B\rangle = \langle (a_1, ..., a_n), (b_1, ..., b_n)\rangle = \sum_{i=1}^{n} a_i b_i$. A subspace L of K^V is said to be *totally isotropic* if $\langle A, B\rangle = 0$ for all $A, B \in L$.

Definition 5.1. A subspace $L \subset K^V$ is an *isotropic system* if it is totally isotropic and of dimension n.

For $X \subset V$ and $A = (a_1, ..., a_n) \in K^V$, denote by $AX = (b_1, ..., b_n)$ the vector defined by $b_i = a_i$ if $i \in X$ and $b_i = 0$ if $i \notin X$. Let $Ai = A\{i\}$. Let $\hat{A} = \{AX | X \subset V\}$ and let $l(A)$ =dim$(L \cap \hat{A})$. A vector A is *complete* if $Ai \neq 0$ for $i = 1, ..., n$. Two complete vectors A, B are *supplementary* if $Ai \neq Bi$, for $i = 1, ..., n$. A complete vector A is an *eulerian vector* of L if $l(A) = 0$. Note that l was originally called the *rank function* of L but this is no longer desirable since Bouchet established the correspondence between isotropic systems and 3-matroids. To avoid ambiguity, the rank function of an isotropic system should coincide with the rank function of the corresponding 3-matroid, as defined by Bouchet and seen in the next subsection.

Theorem 5.2 ([7], (4.1)). *Given an isotropic system $L \subset K^V$ and a complete vector $B \in K^V$, there exists an eulerian vector of L that is supplementary to B.*

Theorem 5.3 ([7], (4.3)(4.4)). *Given an eulerian vector $A = (a_1, ..., a_n)$ of an isotropic system $L \subset K^V$, there exists a unique family $B_1, ..., B_n$ of vectors of L such that $(B_i)i \neq 0$, for $i = 1, ..., n$, and $(B_i)j \in \{0, Aj\}$, for $i \neq j$. Furthermore, $B_1, ..., B_n$ form a basis of L, $0 \neq (B_i)i \neq Ai$, for $i = 1, ..., n$, and $(B_i)j \neq 0 \iff (B_j)i \neq 0$, for $i \neq j$.*

Consider the $n \times n$ matrix M whose rows $B_1, ..., B_n$ form the basis of L corresponding to a given eulerian vector A. Let $F = (V, E)$ be the graph with $[u, v] \in E \iff (B_u)v \neq 0$, for $u \neq v$. Let B be the complete vector whose entries are read from the main diagonal of M, i.e. $B = (B_1)1 + ... + (B_n)n$. It is easy to see that $B_i = AN_F(i) + Bi$, for $i = 1, ..., n$. Therefore L is completely determined by the triple (F, A, B), which is called a *graphic presentation* of L. The converse is also true:

Theorem 5.4 ([7], (3.1)). *Given a triple (F, A, B) where F is a graph on vertex set V and A, B are supplementary vectors of K^V, $L = \{AN_F(X) + BX | X \subset V\}$ is an isotropic system.*

Given an eulerian vector $A = (a_1, ..., a_n)$ of an isotropic system L and $i \in V$, it can be shown that there is exactly one other eulerian vector which differs from A only in i. This vector is denoted by $A * i$ and, given $m = u_1...u_r \in V^*$, $A * m = (...(A * u_1)... * u_r)$.

Theorem 5.5 ([7], (7.1)). *If A and B are two eulerian vectors of an isotropic system L, then there exists a word $m \in V^*$ such that $B = A * m$.*

Theorem 5.6 ([7], (7.6)). *If $P = (F, A, B)$ is a graphic presentation of an isotropic system L, then for every vertex $v \in V$, the graphic presentation associated to the eulerian vector $A * v$ is $P * v = (Fv, A + Bv, B + AN_F(v))$.*

The preceding theorems show that an isotropic system determines a unique Kotzig orbit, that it has essentially the same structure as that orbit and that, conversely, up to the choice of the supplementary vectors A and B, a Kotzig orbit determines a unique isotropic system.

5.2 q-matroids

Consider a partition Ω of a finite set U. Each class $\omega \in \Omega$ is called a *skew class*. Each pair of distinct elements of a skew class is called a *skew pair*. A *subtransversal* (respectively, *transversal*) of Ω is a subset $S \subset U$ such that $|S \cap \omega| \leq 1$ (respectively, $|S \cap \omega| = 1$) holds for all $\omega \in \Omega$. We denote by $\mathcal{S}(\Omega)$ (respectively, $\mathcal{T}(\Omega)$) the set of all subtransversals (respectively, transversals) of Ω.

Definition 5.7. A *multimatroid* is a triple $Q = (U, \Omega, r)$ with a partition Ω of a finite set U and a *rank function* $r : \mathcal{S}(\Omega) \mapsto \mathbb{N}$ satisfying:

- $r(\emptyset) = 0$,
- for $A \in \mathcal{S}(\Omega)$ and $x \in U$ such that A is disjoint from the skew class containing x,

$$r(A) \leq r(A + x) \leq r(A) + 1,$$

- (submodularity inequality) for $A, B \in \mathcal{S}(\Omega)$ such that $A \cup B \in \mathcal{S}(\Omega)$,

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B),$$

- for $A \in \mathcal{S}(\Omega)$ and any skew pair $\{x, y\}$ in a skew class disjoint from A,

$$r(A + x) - r(A) + r(A + y) - r(A) \geq 1.$$

Definition 5.8. A multimatroid Q is a *q-matroid* if all skew classes are of cardinality q.

Definition 5.9. An *independent set* of a multimatroid Q is a subtransversal s such that $r(s) = |s|$. A *base* is a maximal independent set. A *circuit* is a minimal subtransversal that is not independent.

Bouchet shows in [11] how to construct a 3-matroid from an isotropic system L. It is done in the following manner. Let α_i be the vector with α in position i and 0 everywhere else. Define β_i and γ_i in the same manner. Let $\omega_i = \{\alpha_i, \beta_i, \gamma_i\}$, $\Omega = \{\omega_1, ..., \omega_n\}$, $U = \omega_1 \cup ... \cup \omega_n$ and, for $s \in \mathcal{S}(\Omega)$, let $r(s) = |s| - \dim\langle s\rangle \cap L$, where $\langle s\rangle$ denotes the subspace generated by s (or equivalently, since s determines a unique vector of K^V and can be identified with it, $r(s) = |s| - l(s)$). Let $Q(L) = (U, \Omega, r)$.

Theorem 5.10 ([11], Proposition 4.3). *If L is an isotropic system, then $Q(L)$ is a 3-matroid.*

We now show how to construct a 2-matroid from a Sabidussi orbit. One way of deriving this relationship is implicitly given by Bouchet in Section 4 of [6]. However, for convenience, we give a construction similar to the one for 3-matroids.

Let $G = (V, E, c)$ be a bicolored graph of a Sabidussi orbit $\mathcal{O}$. Let A, B be supplementary vectors of K^V and consider the isotropic system L with graphic presentation (F, A, B) where $F = (V, E)$ is the underlying uncolored graph of G. Let $\omega_i = \{Ai, Bi+(1-c(i))Ai)\}$, $\Omega = \{\omega_1, ..., \omega_n\}$ and $U = \omega_1 \cup ... \cup \omega_n$. (For simplicity, we can suppose without altering the structure that $A = (\alpha, ..., \alpha)$ and $B = (b_1, ..., b_n)$ where $b_i = \beta$ if vertex i is white and $b_i = \gamma$ if it is black. In that case, $\omega_i = \{\alpha_i, \gamma_i\}$, for $i = 1, ..., n$.) Let $Q' = Q(L) = (U', \Omega', r')$ be the 3-matroid associated with L. Let $\Omega = \{\omega_1, ..., \omega_n\}$, $U = \omega_1 \cup ... \cup \omega_n$. Since $\mathcal{S}(\Omega)$ is a subset of $\mathcal{S}(\Omega')$, we can define r as the restriction of r' to $\mathcal{S}(\Omega)$. Let $Q(G, A, B) = (U, \Omega, r)$. It follows that $Q(G, A, B)$ is a 2-matroid.

Theorem 5.11. *If $\mathcal{O}$ is a Sabidussi orbit, $G = (V, E, c) \in \mathcal{O}$ and A, B are supplementary vectors of K^V, then $Q(G, A, B)$ is a 2-matroid.*

In the following, let $F = (V, E)$, $G = (V, E, c)$, let A, B be supplementary vectors of K^V and consider the isotropic system L with graphic presentation $P = (F, A, B)$, the 3-matroid $Q' = Q(L) = (U', \Omega', r')$ and the 2-matroid $Q = Q(G, A, B) = (U, \Omega, r)$. As stated by Bouchet ([7] (8.3)), if $[v, w] \in E$, then $P * vwv = (Fvwv, A + x, B + x)$, where $x = A\{v, w\} + B\{v, w\}$. We identify the vectors of K^V with the subtransversals of Q' in the natural manner.

Proposition 5.12. *If u is a white vertex of G, then $A * u \in \mathcal{S}(\Omega)$. If v, w are adjacent black vertices of G, then $A * uvu \in \mathcal{S}(\Omega)$.*

Proof. We use Theorem 5.6 and the previous remark. For $i \neq u$, $(A * u)i = Ai \in \omega_i$. For $i = u$, $(A * u)i = Ai + Bi \in \omega_i$. For $v \neq i \neq w$, $(A * vwv)i = Ai \in \omega_i$. For $i \in \{v, w\}$, $(A * vwv)i = Ai + (Ai + Bi) = Bi \in \omega_i$. □

It follows that the complementation sets of G are in bijection with the eulerian vectors of L that are contained in $\mathcal{S}(\Omega)$. To be precise, if A' is an eulerian vector of L in $\mathcal{S}(\Omega)$ with corresponding graphic presentation (F', A', B'), let $S = \{i \in V | Ai \neq A'i\}$, $V' = \{i \in V | A'i + B'i \in \omega_i\}$ and let G' be the bicolored graph with underlying graph F and with u white if and only if $u \in V'$. Then $G' = GS$ and $Q(G', A', B') = Q(G, A, B)$. Therefore, up to the choice of the supplementary vectors A and B, Sabidussi orbits can be identified with a subset of the 2-matroids. Call this subset the *graphic 2-matroids*.

There exist non-graphic 2-matroids. As an example, consider the 2-matroid $Q = (U, \Omega, r)$ with $U = \{a, a', b, b', c, c'\}$, $\Omega = \{\omega_a = \{a, a'\}, \omega_b = \{b, b'\}, \omega_c = \{c, c'\}\}$ and r determined by the set $\mathcal{C}(Q) = \{\{a, b'\}, \{a', b\}, \{a, b, c'\}\}$ of the circuits of Q. To show that Q is not graphic, we proceed by contradiction. Suppose that Q is graphic with an associated isotropic system $L \subset K^V$, $V = \{a, b, c\}$, and a bicolored graph G corresponding to the eulerian vector $\{a, b, c\}$ of L. Since $\{a, b', c\}$ and $\{a', b, c\}$ are not independent but $\{a', b', c\}$ is, vertices a and b must be adjacent black vertices of G. Since $\{a', b', c'\}$ is independent, c must be white in $G[ab]$, hence it is white in G. On the other hand, $\{a, b, c'\}$ is not independent, so c must be black in G, a contradiction.

Definition 5.13. A *set system* is a pair $S = (X, \mathcal{F})$ where X is a finite set and $\mathcal{F}$ is a family of subsets of X, called the *feasible sets* of S. A *delta-matroid* is a set system $(X, \mathcal{F})$ with $\mathcal{F} \neq \emptyset$ and satisfying the following *symmetric exchange axiom.*

$$F_1, F_2 \in \mathcal{F} \text{ and } x \in F_1 + F_2 \Rightarrow \exists y \in F_1 + F_2 \text{ such that } F_1 + \{x, y\} \in \mathcal{F}$$

Definition 5.14. Given a 2-matroid $Q = (U, \Omega, r)$ and a transversal X of Ω, the set system $Q \cap X = (X, \mathcal{F})$, where $\mathcal{F} = \{A \cap X | A \text{ is a base of } Q\}$, is called the *trace* of Q on X.

Theorem 5.15 ([11], Proposition 4.2). *A set system is a delta-matroid if and only if it is equal to the trace of a 2-matroid on one of its transversals.*

From the correspondence between Sabidussi orbits and the graphic 2-matroids, we get the following corollary.

Corollary 5.16. *For a given bicolored graph $G = (V, E, c)$, the set system $(V, \mathcal{F})$, where $\mathcal{F}$ is the family of complementation sets of G, is a delta-matroid.*

Proof. Let $V = \{1, ..., n\}$, $A = (\alpha, ..., \alpha)$ and $B = (b_1, ..., b_n)$ where $b_i = \beta$ if vertex i is white and $b_i = \gamma$ if it is black. Consider the trace of the 2-matroid $Q(G, A, B)$ on $X = \{\gamma_1, ..., \gamma_n\}$. The result follows. □

Proposition 5.17. *The maximal feasible sets of a delta-matroid are equicardinal.*

Proof. By contradiction, suppose that F_1, F_2 are maximal feasible sets of different sizes chosen so that $|F_1 + F_2|$ is minimal. Without loss of generality, let $x \in F_2 \backslash F_1$ and $y \in F_1 + F_2$ such that $F_1 + \{x, y\}$ is a feasible set. By the maximality of F_1, $y \notin F_2$, so that $|F_1 + \{x, y\}| = |F_1|$ but $|(F_1 + \{x, y\}) + F_2| = |F_1 + F_2| - 2$, contradicting the minimality of $|F_1 + F_2|$. □

Although a direct proof of the following non-trivial result is certainly possible, it is an immediate consequence of the fact that the family of the complementation sets of a bicolored graph has the structure of a delta-matroid.

Corollary 5.18. *For a given bicolored graph G, all maximal complementation sets have the same cardinality.*

6 Application to the vertex-nullity polynomial

In [3], Arratia *et al.* define the vertex-nullity polynomial of a bicolored graph (in their paper, a bicolored graph is called a *looped graph*). For $S \subset V$, let $G[S]$ be the bicolored subgraph of G induced by S. Let the adjacency matrix of a bicolored graph G encode the coloring on its diagonal so that position uu holds $1 - c(u)$. (Or, equivalently, consider white vertices as having loops.) Let $n(G)$ be the nullity (corank) of the adjacency matrix of G.

Definition 6.1. The *vertex-nullity polynomial* of a bicolored graph $G = (V, E, c)$ is

$$q_N(G; x) = \sum_{S \subset V} (x-1)^{n(G[S])}.$$

Theorem 6.2 ([3], Theorem 6). *Let u, v be adjacent black vertices of $G = (V, E, c)$. Then*

$$q_N(G) = q_N(G - u) + q_N(G[uv] - u).$$

Let u be white in $G = (V, E, c)$. Then

$$q_N(G) = q_N(G - u) + q_N(Gu - u).$$

From the preceding theorem and Theorem 4.3, it is easy to see that the vertex-nullity polynomial of a bicolored graph is invariant over its Sabidussi orbit. Therefore, the vertex-nullity polynomial of a Sabidussi orbit is defined in the following manner.

Definition 6.3. The *vertex-nullity polynomial* of a Sabidussi orbit $\mathcal{O}$ containing a bicolored graph G is

$$q(\mathcal{O}) = q_N(G).$$

We can define a vertex-nullity polynomial for Kotzig orbits as well. Aigner and van der Holst [1] showed that the following graph polynomial is invariant over its Kotzig orbit. For $S \subset V$, let $G[S]$ be the subgraph induced by S.

Definition 6.4. The *vertex-nullity polynomial* $Q(G, x)$ of a simple graph G is

$$Q(G; x) = \sum_{S \subset V} (x-2)^{n(G[S])}.$$

Definition 6.5. The *vertex-nullity polynomial* of a Kotzig orbit $\mathcal{O}$ containing a graph G is

$$q(\mathcal{O}) = Q(G).$$

Aigner *et al.* [1] established the relationship between the interlace polynomials presented in this section and the Tutte-Martin polynomials introduced by Bouchet in [9]. We can now interpret their result as follows : the complementation polynomials of Kotzig and Sabidussi orbits are equivalent to the Tutte-Martin polynomials of the corresponding isotropic systems and 2-matroids, respectively.

7 Conclusion

In order to study local complementation in a systematic manner, we found that the concept of substitution rules is fundamental. We have characterized completely the local substitution rules. The insight gained into the structure of Sabidussi orbits enabled us to establish the relationship between Sabidussi orbits and 2-matroids, thereby generalizing the 2-matroids that are obtainable from 4-regular connected graphs with a given transition system (see [6, 11, 16]). The reader is referred to [16] where the link between Sabidussi orbits and the Cycle Double Cover Conjecture and Sabidussi's Compatibility Conjecture is explained. In light of the work presented here, it would be interesting to see what the implications of multimatroid theory are for these two important conjectures.

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