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SOME b-CONTINUOUS CLASSES OF GRAPH

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Some b-continuous classes of graph

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Abstract

In this paper we are interested in the b-chromatic coloring of a graph.

Some graphs have a b-chromatic p-coloring and a b-chromatic q-coloring with $p < q$, but no r-coloring which is b-chromatic with $p < r < q$. Otherwise, the graph is called b-continuous.

We prove that the hypercube H_n ($n \neq 3$), trees and apart from some exceptions, the 3-regular graphs are b-continuous.

Keywords : Graph algorithms; b-chromatic coloring; b-continuous graphs; Trees.
AMS : 05C15

Résumé

Dans cet article, on s'interesse à la coloration b-chromatique d'un graphe.

Certains graphes possèdent une p-coloration b-chromatique et une q-coloration b-chromatique avec $p < q$, mais il n'existe pas de r-coloration b-chromatique avec $p < r < q$. Dans le cas contraire on dira que le graphe est b-continu.

On prouve que l'hypercube H_n ($n \neq 3$), les arbres, et à part quelques exceptions les graphes cubiques sont b-continus.

Mot-clé : Algorithmes de graphes; Coloration b-chromatique; Arbres.
AMS : 05C15

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A vertex coloring of G consists in assigning to each vertex of G a color in such way that no two adjacent vertices have the same color. If k colors are used, the result is called a k-coloring of G. The chromatic number $\chi(G)$ is the minimum integer k for which G has a k-coloring.

We call b-chromatic a k-coloring of G such that for every color c there exists at least one vertex of color c adjacent to a vertex of every other color. Such a vertex is called a b-chromatic vertex of c. The b-chromatic number $\varphi(G)$ is the maximum k for which G has a b-chromatic k-coloring. This parameter was first studied by Irving and Manlove [1].

A k-coloring of a graph such that for each pair of different colors there are two adjacent vertices with these colors is called a complete or achromatic coloring. The achromatic number $\psi(G)$ of G is the maximum k for which G has an achromatic k-coloring. Harary, Hedetniemi and Prins [3] proved that for each graph G and each k with $\chi(G) \leq k \leq \psi(G)$, there is an achromatic k-coloring of G. Christen and Selkow [4] proved that similar property holds for the Grundy coloring. A Grundy k-coloring of G is a k-coloring of G using colors $c_1, \ldots, c_k$ such that every vertex colored c_i, for each $0 \leq i \leq k$, is adjacent to at least one vertex colored c_j, for each $1 \leq j < i$.

In contrast with the Grundy and achromatic colorings, some graphs have a b-chromatic p-coloring and a b-chromatic q-coloring with $p < q$, but no r-coloring which is chromatic with $p < r < q$. A graph is said to be b-continuous if it has a b-chromatic k-coloring for any k, with $\chi(G) \leq k \leq \varphi(G)$. The question of knowing which graphs are b-continuous remains open for general graphs.

A graph is called $\psi\chi$-perfect if for each induced subgraph H of the graph $\chi(H) = \psi(H)$. In [4] they characterize the class of $\psi\chi$-perfect graphs. On the other hand, we have $\varphi(G) \leq \psi(G)$, for any graph G. So, if a graph G is $\psi\chi$-perfect then $\chi(G) = \varphi(G)$. Hence the $\psi\chi$-perfect graphs are b-continuous.

In [5], Kratochvil, Tuza and Voigt characterize the graphs having b-chromatic number 2, such graphs are b-continuous, they proved also that that for every n, the complete bipartite graph $K_{n,n}$ removing a perfect matching has a b-chromatic coloring by k colors if and only if $k = 2$ or $k = n$. This give us an infinite family of non b-continuous graphs.

In this paper we show that the hypercube H_n with $n \neq 3$, the trees and apart some exceptions the 3-regular graphs are b-continous.

2 The b-continuity of the hypercube

We denoted by H_n the hypercube of dimension n. In [1, 5] they proved that H_3 has b-chromatic 2-coloring and b-chromatic 4-coloring,but there is no 3-coloring of H_3 that is b-chromatic. Apart from H_3 we will show that for every $n \neq 3$, the hypercube H_n is b-continuous. From the corollary 2.1 We can deduce that $\varphi(H_{n+1}) = \varphi(H_n) + 1$.

Corollary 2.1 *[2] We have $\varphi(H_1) = \varphi(H_2) = 2$ and $\varphi(H_n) = n + 1$, for all $n \geq 3$.*

Theorem 2.1 *For every n, $n \neq 3$ the hypercube H_n is b-continuous.*

Proof. Obviously, H_1 and H_2 are b-continuous. For $n \geq 4$ we proof the required property by induction on n.

We have $\chi(H_n) = 2$ and $\varphi(H_n) = n+1$ for all n. In particular $\chi(H_4) = 2$ and $\varphi(H_4) = 5$. Figure 1 presents a b-chromatic 3-coloring and b-chromatic 4-coloring of H_4, so H_4 is b-continous. (In this figure the black nodes denote the b-chromatic vertices).

Induction hypothesis : assume that H_n is b-continuous.

It is well known that $H_{n+1} = H_n \Box K_2$, which means that H_{n+1} can be viewed as two copies H_n^1, H_n^2 of the hypercube of dimension n such that : if $x_1^1, \dots, x_{2^n}^1$ are the vertices of H_n^1 and $x_1^2, \dots, x_{2^n}^2$ are the vertices of H_n^2, there is an edge between x_i^1 and x_i^2.

By induction, for every p, $2 \leq p \leq \varphi(H_n)$, H_n has a b-chromatic p-coloring. Let C be a b-chromatic coloring of H_n^1 and denote by $c_1, \dots, c_p$ the colors used by C. Let $c_1', \dots, c_p'$ be a derangement of the colors $c_1, \dots, c_p$. For every i, $1 \leq i \leq 2^n$, assign the color c_j' to the vertex x_i^2 if the vertex x_i^1 is colored c_j. As $c_1', \dots, c_p'$ is a derangement of $c_1, \dots, c_p$, it is straigthforward to verify that the resulting coloring is a b-chromatic p-coloring of H_{n+1}. Hence H_{n+1} does have a b-chromatic p-coloring for every p, $2 \leq p \leq \varphi(H_n)$ and as $\varphi(H_{n+1}) = \varphi(H_n)+1$, H_{n+1} is b-continuous. □

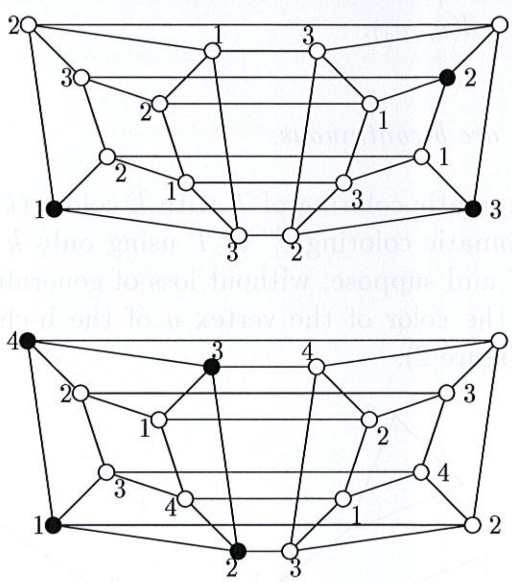


Fig. 1: A b-chromatic 3 and 4-coloring of H_4

3 The b-continuity of trees

In this section, we prove that the trees are b-continuous, our method hinge on reducing a b-chromatic k-coloring to a b-chromatic $(k-1)$-coloring. We now define a special vertex which we call an extreme vertex.

Definition 3.1 *Let $T = (V, E)$ be a tree, and let C be a b-chromatic coloring of T. Assume that $v \in V$ is a b-chromatic vertex of C. Then v is an extreme vertex of C, if the forest T_v*

induced by $V \setminus \{v\}$ contains exactly one subtree T_v^b which we call the b-chromatic subtree of C, such that T_v^b contains all the other b-chromatic vertices of C.

The following lemma ensures the existence of an extreme vertex.

Lemma 3.1 *Let C be a b-chromatic coloring of $T = (V, E)$. Then there exists at least two extreme vertices of C.*

Proof. Let $B_C = \{v_1, v_2, \dots, v_m\}$ be the set of the b-chromatic vertices of T with respect to C, and let D_C^b be the b-chromatic diameter of T with respect to C, defined by

$$D_C^b = \max_{v_i, v_j \in B_C} d(v_i, v_j).$$

Suppose that $D_C^b = d(v_r, v_q)$, then v_r and v_q are extreme vertices. If not, suppose for example that v_r is not an extreme vertex, then the forest T_{v_r} contains at least two trees $T_{v_r}^1$ and $T_{v_r}^2$, such that $T_{v_r}^1$ and $T_{v_r}^2$ contains b-chromatic vertices. Suppose that $T_{v_r}^1$ contains the vertex v_q, then the path in T from v_q to any b-chromatic vertex belonging to $T_{v_r}^2$ pass through v_r, a contradiction with $D_C^b = d(v_r, v_q)$. □

Theorem 3.1 *The trees are b-continuous.*

Proof. Let C be a b-chromatic coloring of T with k colors ($k \geq 3$). We will show that we can reduce C to a b-chromatic coloring C' of T using only $k-1$ colors. For this, choose an extreme vertex v of C and suppose, without loss of generality, that c_1 is the color of the vertex v, and that c_2 is the color of the vertex u of the b-chromatic subtree T_v^b which is adjacent to v in T (see Figure 2).

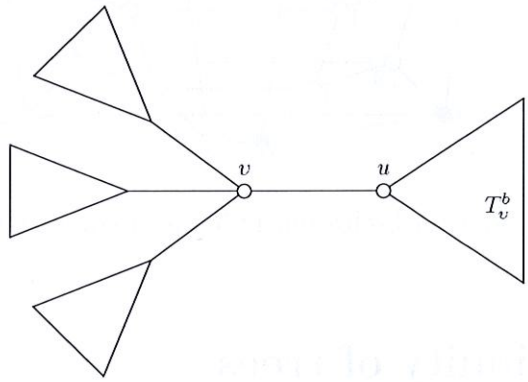


Fig. 2: v is an extreme vertices of T and T_v^b is the b-chromatic subtree of T

As T_v is a forest, we may easily recolor all the trees of $T_v \setminus T_v^b$ with c_2 and c_3 such that all adjacent vertices to v in T has the color c_2. We consider two cases.

Case 1 : Vertex v was the unique b-chromatic vertex for the color c_1, then the color c_1 has lost his unique b-chromatic vertex. Hence for each vertex w colored c_1, not all of colors

$c_2, c_3, \ldots, c_k$ appear on the neigbhors of w. So, it is possible to recolor each w of the color c_1 (including v) using the a missing color in the neigbhood of w. Then, we would terminate with the desired coloring C'.

Case 2 : Vertex v wasn not the unique b-chromatic vertex for the color c_1. In this case we choose an extreme vertex of the new coloring and we iterate our recoloring process. It is straightforward to verify that we loose one and only one b-chromatic vertex each time we apply our recoloring process. Hence, it turns out that after a finite number of steps, one color must loose all its b-chromatic vertices, so this case reduces to the previous case.

Since we can reduce each b-chromatic coloring of size k to a b-chromatic coloring of size $k-1$, for all k, $3 \leq k \leq \varphi(T)$, and since the chromatic coloring is a b-chromatic coloring, it follows that for each k between the b-chromatic number and the chromatic number, T has a b-chromatic coloring of size k. □

Corollary 3.1 *If T is a tree, then for any $k \leq \varphi(T)$, a b-chromatic k-coloring of T is polynomial-time computable.*

Proof. A polynomial-time algorithm for constructing maximum b-chromatic coloring for trees was given in [1]. On the other hand, the proof of Theorem 3.1 induces a polynomial-time algorithm for reducing any b-chromatic p-coloring to a b-chromatic $(p-1)$-coloring. Hence we can obtain one b-chromatic k-coloring, for $k \leq \varphi(T)$ in polynomial-time. □

4 The b-continuity of 3-regular graphs

Notation 4.1 *We denote by $\overline{C}_{10}$ (Figure 3 the cycle C_{10} with all its chords of length 5.*

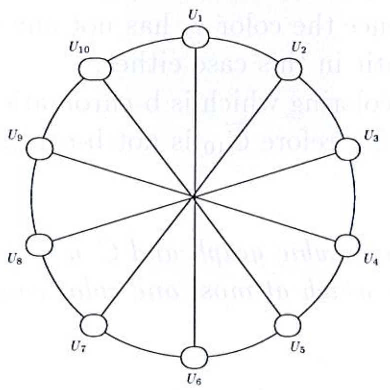


Fig. 3: $\overline{C}_{10}$ graph

Proposition 4.1 *The graph $\overline{C}_{10}$ is not b-continuous.*

Proof. $\overline{C}_{10}$ is a bicubic graph, (bipartite 3-regular graph), so $\overline{C}_{10}$ has a b-chromatic 2-coloring. A b-chromatic 4-coloring of $\overline{C}_{10}$ is given by coloring the vertices u_1, u_4, u_8 by c_1, the vertices u_7, u_{10} by c_2, the verices u_2, u_5 by c_3 and u_3, u_6, u_9 by c_4. We will show that there is no b-chromatic 3-coloring of $\overline{C}_{10}$. Assume the opposite and let show that it laeds to a contradiction. We denote by c_1, c_2, c_3 the colors used by C.

Case 1 : Suppose that there exists a b-chromatic vertex of C, such that its neighbors in C_{10} are of the same color. By symmetry we can suppose that u_1 is this vertex. Suppose that c_1 is the color of u_1 and c_2 the color of its two neighbors in C_{10}. Since u_1 is b-chromatic, u_6 must be colored c_3. This forces u_5 and u_7 to be colored by c_1.

Suppose that u_2 is b-chromatic for the color c_2, then u_3 should be colored by c_3, which implies that the vertices u_4 and u_8 should be colored by c_2. Neither u_3 nor u_6 is b-chromatic for the color c_3. All the adjacent vertices of the last uncolored vertex have the same color : so it can't be b-chromatic vertex for the color c_3. Hence u_2 cannot be b-chromatic for the color c_2. By symmetry the vertex u_{10} cannot be b-chromatic for c_2. So the vertices u_3 and u_9 should be colored by c_1. This implies that all the adjacent vertices of the uncolored vertices have the same color, so the uncolored vertices cannot be b-chromatic for the color c_2. It follows that the color c_2 hasn't any b-chromatic vertex. So there is no 3-coloring which is b-chromatic in this case.

Case 2 : There is no b-chromatic vertex of C, such that its neighbors in C_{10} are of the same color. Suppose that u_1 is b-chromatic for the color c_1, and let u_{10} be of color c_2 and u_2 of color c_3. By symmetry we can suppose that u_6 is colored by c_2. This implies that u_7 must be colored by c_1. Vertex u_7 is b-chromatic for the color c_1 and u_6 is colored by c_2, so by hypothesis u_8 must be colored by c_3, which implies that u_9 must be colored by c_1. The vertices adjacent to u_8 in C_{10} are of the same color, hence by hypothesis u_8 is not a b-chromatic vertex, so u_3 must be colored by c_1. Similarly, u_{10} is not a b-chromatic vertex of the color c_2 and u_5 must be colored by c_1. At this time no vertex colored c_2 is b-chromatic for this color and the last vertex u_4 cannot be a b-chromatic for the color c_2, because u_3 and u_5 are colored by c_1. Hence the color c_2 has not any b-chromatic vertex. So there is no 3-coloring which is b-chromatic in this case either.

$\overline{C}_{10}$ does not have any 3-coloring which is b-chromatic, but it has a b-chromatic coloring of size respectively 2 and 4. Therefore $\overline{C}_{10}$ is not b-continuous. □

Proposition 4.2 *Let G be a bicubic graph and C a b-chromatic 3-coloring of G. If H is a connected component of G in which at most one color can have b-chromatic vertices, then H is isomorphic to $K_{3,3}$.*

Proof. Suppose that H is a connected component of G such that $H \neq K_{3,3}$. Let U, V be the two classes of its bipartition. First assign color c_1 to U and c_2 to V. Since H is connected there exists an edge $[u_1, v_1]$ with $u_1 \in U$, $v_1 \in V$. Let u_2, u_3 be the other neighbors of v_1 in U and v_2, v_3 the other neighbors of u_1 in V. Since $H \neq K_{3,3}$ at least one of the edges $[u_2, v_2], [u_2, v_3], [u_3, v_2], [u_3, v_3]$ is missing, for instance $[u_2, v_2]$. In this case recoloring u_2 and v_2 by c_3 makes u_1 and v_1 b-chromatic.

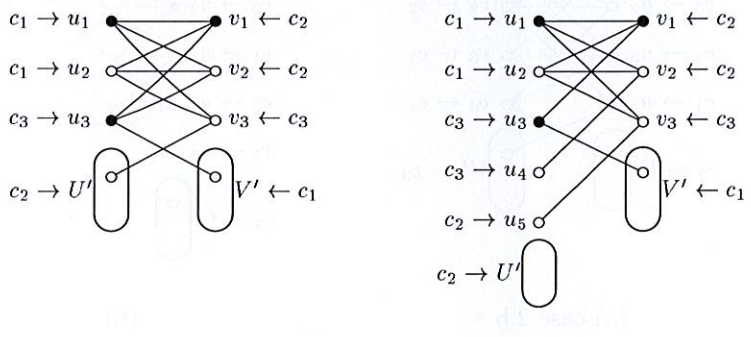
Therefore it remains to prove that $K_{3,3}$ cannot contain b-chromatic vertices for more than one color. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ be the two classes of bipartition. Suppose that u_1 is b-chromatic for c_1, this implies that the vertices u_2 and u_3 are also adjacent to the colors c_2 and c_3, so u_2 and u_3 must be colored by c_1, which means that the vertices v_1, v_2 and v_3 have only the color c_1 in their neighborhood, so they can't be b-chromatic vertices. □

Theorem 4.1 *Apart from the cube H_3 and the graph $\overline{C}_{10}$ any 3-regular graph is b-continuous*

Notation 4.2 *In the following figures ($c_i \to v_j$) means that we assign the color c_i to the vertex v_j, ($c_i \to U'$) means that we assign the color c_i to the vertices of the set U', and the black vertices denote the b-chromatic vertices.*

Proof. Let G be a 3-regular non bipartite graph. Clearly we have $3 \leq \chi(G) \leq \varphi(G) \leq \Delta(G) + 1 = 4$, hence G is b-continuous.

Let G be a bicubic graph. G is b-continuous if it has a b-chromatic 3-coloring or if $\chi(G) = \varphi(G) = 2$. Denote by c_1, c_2 and c_3 the colors used by a b-chromatic 3-coloring of G if such a coloring exists.



(a) case 1.a

(b) case 1.b

Fig. 4: Case 1

• If G contains at least 3 connected components $H_1, H_2, H_3, \ldots$, then a b-chromatic 3-coloring of G can be easily obtained by coloring the connected component H_i, for $1 \leq i \leq 3$ in such way that it contains a b-chromatic vertex for the color c_i. As G is bipartite each other connected component can be colored by 2 colors. Hence G does have a b-chromatic 3-coloring, so G is b-continuous.

• G has 2 connected components H_1 and H_2.

1- If at least one connected component, for example H_1 is not the complete bicubic graph $K_{3,3}$, then by Proposition 4.2, we can give a coloring of H_1 such that each color c_1 and c_2 has a b-chromatic vertex. And we color H_2 in order to obtain a b-chromatic vertex for c_3. This gives a b-chromatic 3-coloring of G. Hence G is b-continuous.

2- If $H_1 = H_2 = K_{3,3}$ then $\chi(G) = \varphi(G) = 2$. So G is b-continuous.

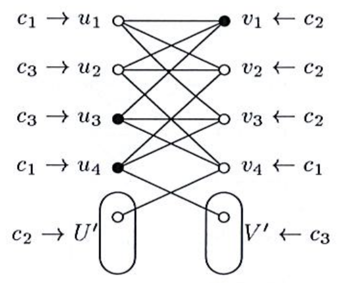
• It remains to study the case of a connected bicubic graph $G = (U \cup V, E)$. If $G = K_{3,3}$ then $\chi(G) = \varphi(T) = 2$. So G is b-continuous. Henceforth we consider that $G \neq K_{3,caslaa.pscaslaa.ps3}$.

Case 1 : There exist two vertices, say u_1 and u_2, having the same neighborhood $\{v_1, v_2, v_3\}$.

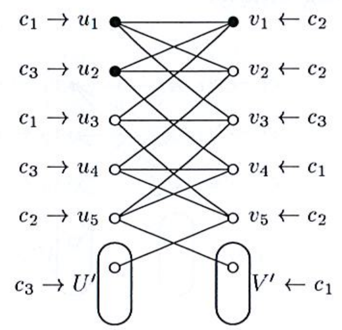
Case 1.a : Two vertices among v_1, v_2, v_3, say v_1, v_2, have a common third neighbor u_3. As $G \neq K_{3,3}$, there is no edge between u_3 and v_3. Hence the sets $U' = U \setminus \{u_1, u_2, u_3\}$ and $V' = V \setminus \{v_1, v_2, v_3\}$ are not empty. Figure 4(a) illustrates a b-chromatic 3-coloring of G.

Case 1.b : The vertices v_1, v_2, v_3 have distinct third neighbors, u_3, u_4, u_5 respectively. Then Figure 4(b) illustrates a b-chromatic 3-coloring of G.

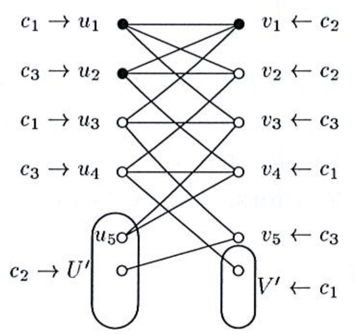
Case 2 : Any two vertices have at most two neighbors in common, and there exist two vertices u_1, u_2 having two common neighbors v_1, v_2. Let v_3, v_4 be respectively the third neighbors of u_1, u_2, and let u_3, u_4 be respectively the third neighbors of v_1, v_2. We have $v_3 \neq v_4$, otherwise the vertices u_1 and u_2 will have three common neighbors. Similarly we have $u_3 \neq u_4$. We have five subcases according to the structure of the graph induced by the set u_3, u_4, v_3, v_4.



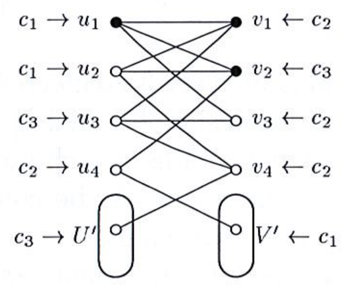
(a) case 2.b



(b)

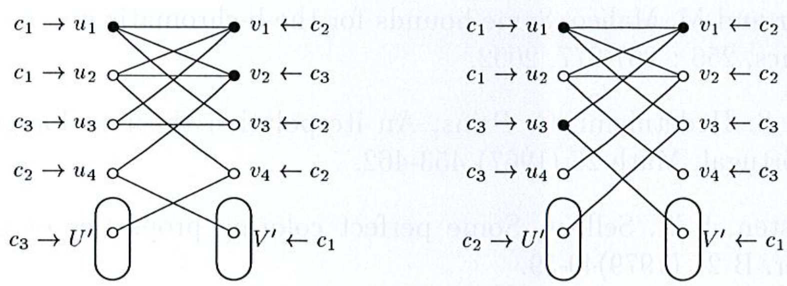


(c)



(d) case 2.c.2

Case 2.a : The graph induced by u_3, u_4, v_3, v_4 is complete. In this case the graph G is the cube H_3, which is excluded.



(e) cas 2.d (f) cas 2.e

Figure 5: Case 2

Case 2.b : The induced graph contains three edges, and by symmetry we can suppose that the missing edge is $[u_4, v_4]$. In this case a b-chromatic 3-coloring is shown in Figure 5(a).

Case 2.c : The induced graph by u_3, u_4, v_3, v_4 contains two edges.

Case 2.c.1 : These two edges compose a matching, by symmetry we may suppose that the matching is $\{[u_3, v_3], [u_4, v_4]\}$.

$*$ If $|G| = 10$, then G is the graph $\overline{C}_{10}$ (we have to add edges $[u_4, v_5], [u_4, v_5], [u_5, v_5]$ in order to saturate the graph). By Proposition 4.1, G is not b-continuous.

$*$ If $|G| \geq 12$ and $[u_4, v_5], [u_5, v_4] \in E$, then we must have $[u_5, v_5] \notin E$. Figure 5(b) shows a b-chromatic 3-coloring of G. Hence the graph G is b-continuous.

$*|G| \geq 12$ and at least one edge between $[u_4, v_5]$ and $[u_5, v_4]$ is missing, by symmetry we may suppose that the missing edge is $[u_4, v_5]$. Figure 5(c) give a b-chromatic 3-coloring in this case. Hence G is b-continuous.

Case 2.c.2 : The induced graph by u_3, u_4, v_3 and v_4 contains two adjacent edges, say $[u_3, v_3]$ and $[u_3, v_4]$. Figure 5(d) gives a b-chromatic 3-coloring.

Case 2.d : There is just one edge in the induced graph by u_3, u_4, v_3, v_4, say $[u_3, v_3]$. In this case a b-chromatic 3-coloring of G is shown in Figure 5(e).

Case 2.e : There is no edge in the graph induced by u_3, u_4, v_3, v_4. In this case a b-chromatic 3-coloring of G is shown in Figure 5(f).

Case 3 : Any two vertices have at most one neighbor in common. Let u_1 be a vertex of U and v_1, v_2, v_3 its neighbors, let u_2, u_3 be the other neighbors of v_1 in U, u_4, u_5 be the other neighbors of v_2 in U and u_6, u_7 be the other neighbors of v_3 in U. Vertices v_1, v_2, v_3 have u_1 as a common neighbor, so for each i, j, $2 \leq i < j \leq 7$, $u_i \neq u_j$. A b-chromatic 3-coloring of G is the following one : assign the color c_1 to u_1, c_2 to u_3, u_4, u_5, u_6, u_7 and v_1, c_3 to v_2, v_3 and u_2, u_3, assign c_1 to all uncoloured vertices in V and c_2 to all uncolored vertices in U. G has a b-chromatic 3-coloring so G is b-continuous. □

References

[1] R. W. Irving and D. F, Manlove. The b-chromatic number of a graph. Discrete Applied Mathematics, 91 : 127-141,1999.

[2] M. Kouider and M. Mahéo, Some bounds for the b-chromatic number of graph. Discrete Mathematics, 256 : 267-277, 2002.

[3] F. Harary, S. Hedetniemi, G. Prins, An iterpolation theorem for graphical homomorphisms, Portugal. Math 26 (1967) 453-462.

[4] C.A. Christen, S.M. Selkow, Some perfect coloring properties of graphs, J. Combin. Theory, Ser. B 27 (1979)49-59.

[5] J. Kratochvil, , Z. Tuza, M. Voigt, On the b-chromatic number of a graphs, WG 2002, LNCS 2573, 2002 310-320.

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