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HAGGKVIST R / LI H

Unité Mixte de Recherche 8623
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CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 650
91405 ORSAY Cedex (France)

Long cycles in graphs with some large degree vertices

Roland Häggkvist

Department of Mathematics
University of Umeå
S-90187 Umeå
e-mail: rolandh@abel.math.umu.se

Hao LI

Laboratoire de Recherche en Informatique
UMR 8623, C.N.R.S.-Université de Paris-sud
91405-Orsay cedex, FRANCE
e-mail: li@lri.fr

Abstract

Let G be a graph of order n and k an integer with $3 \leq k \leq n-1$. We obtain that if there are at least $n/2-1$ vertices of degree at least k then either the circumference of G is at least k or G has a subgraph isomophic to the graph obtained from $K_{\frac{k-1}{2},\frac{k+3}{2}}$ by adding an edge between any pair of vertices in the $\frac{k-1}{2}$-vertex-part. (Hence the circumference of G is at least $k-1$). By using above result, we show that the following conjecture of Woodall is true if the graphe is 3-connected and $k \geq 25$: if a 2-connected graph of order n has at least $\frac{n}{2}+k$ vertices of degree at least k, then it has a cycle of length at least $2k$. This conjecture was one of the 50 unsolved problems in [2].

1 Introduction and notation

All the graphs considered in this paper are undirected and simple. We use the notation and terminology in [2]. In addition, for a graph $G=(V(G),E(G))$, let H be a subgraph of G. Then the neighborhood in H of a vertex $u \in V(G)$ is $N_H(u)=\{v \in V(H) \ : \ uv \in E(G)\}$ and the degree of u in H is $d_H(u)=|N_H(u)|$. The minimum degree in G of the vertices in H is denoted by $\delta(H)$. If $X \subseteq V(G)$, let $N_H(X)=\cup_{v\in X}(N_H(v)-X)$. In the case $H=G$, we use $N(u), d(u)$, δ and $N(X)$ instead of $N_G(u), d_G(u)$, $\delta(G)$ and $N_G(X)$, respectively.

If $C=c_1c_2...c_pc_1$ is a cycle, we let $C[c_i,c_j]$, for $i \leq j$, be the subpath $c_ic_{i+1}...c_j$, and $\overline{C}[c_j,c_i]=c_jc_{j-1}...c_i$, where the indices are taken modulo p. We will consider $C[c_i,c_j]$ and $\overline{C}[c_j,c_i]$ both as paths and as vertex-sets. Define $C(c_i,c_j]=C[c_{i+1},c_j]$, $C[c_i,c_j)=C[c_i,c_{j-1}]$ and $C(c_i,c_j)=C[c_{i+1},c_{j-1}]$. For any i, we put $c_i^+=c_{i+1}$, $c_i^-=c_{i-1}$, and for any $j \geq 2$, $c_i^{+j}=c_{i+j}$ and $c_i^{-j}=c_{i-j}$. For $A \subseteq C$, we set $A^+=\{v^+|v \in A\}, A^-=\{v^-|v\in A\}$, for any $j \geq 2$, $A^{+j} = \{v^{+j} | v \in A\}$ and $A^{-j} = \{v^{-j} | v \in A\}$. We will use similar definitions for a path.

We denote by $c(G)$ the circumference, i.e. the length of a longest cycle in G.

Various longest cycle problems are interesting and important in basic graph theory and have been deeply studied. The main problem studied in this paper is the circumferences of graphs. A classical result is due to Dirac

Theorem 1 [3] : If G is a 2-connected graph on $n \geq 3$ vertices, then $c(G) \geq \min\{n, 2\delta\}$.

The above results based on conditions on degrees of all vertices of the graph. It is natural to ask if we can still get a long cycle when the graph contains many vertices of large degrees. We obtain the followings:

Theorem 2: Let G be a graph of order n and k an integer with $3 \leq k \leq n-1$. If there are at least $n/2 - 1$ vertices of degree at least k then either the circumference of G is at least k or G has a subgraph isomophic to the graph $K^*_{\frac{k-1}{2}, \frac{k+3}{2}}$ which is obtained from the complete bipartite graph $K_{\frac{k-1}{2}, \frac{k+3}{2}}$ by adding an edge between any pair of vertices in the $\frac{k-1}{2}$–vertex-part.. (Hence the circumference of G is at least $k-1$).

The following examples are interesting. Let $K^*_{\frac{k-1}{2}, \frac{k+3}{2}} := D(X, Y)$ with $|X| = \frac{k-1}{2}$ and $|Y| = \frac{k+3}{2}$. Pick up q copies $D_i(X_i, Y_i), 1 \leq i \leq q$, of $D(X, Y)$ and let $u_i, v_i \in Y_i$. Denote by H the graph obtained by identifying v_i and u_{i+1} for $1 \leq i \leq q-1$. Then H has $q(k-1)/2 + q - 1$ vertices of degree at least k and we have $q(k-1)/2 + q - 1 = \frac{1}{2}(q(k-1)/2 + q(\frac{k+3}{2} - 1) + 1) + \frac{q-3}{2}$. These examples show that the circumference may be less than k even if the number of vertices of degree at least k is at least $n/2 + c$ for any fixed c.

As an improvement of Dirac's theorems, Woodall made the following conjecture in 1975: If a 2-connected graph of order n has at least $\frac{n}{2} + k$ vertices of degree at least k, then it has a cycle of length at least $2k$. This conjecture was one of the 50 unsolved problems in the book [2] and has been essentially proved in [5]. But we give a proof of the followings by using Theorem 2.

Theorem 3: If $k \geq 25$ and a 3-connected graph of order n has at least $\frac{n}{2} + k$ vertices of degree at least k, then it has a cycle of length at least $2k$.

2 Preliminary lemmas

Lemma 1: Let $G = (V, E)$ be any 2-connected graph and $B := \{v : d(v) \geq k-1\}$, $3 \leq k \leq n/2$. If $S := G - B$ is independent and if for any set $X \subseteq S$ with a common neighbor (*i.e.*, $X \subseteq S \cap N(x)$ for some $x \in B$),

$$|N(X)| > \frac{|X|}{2},$$

then G has a cycle of length at least $min\{|B|, k\}$.

Proof of Lemma 1 : Here we just give a proof for existence of a cycle with at least $min\{|B|, k-3\}$ vertices. A detailed proof of the lemma can be found in Appendix.

Let $P := v_1 v_2 ... v_p$ be a path in G such that

(a) $v_1, v_p \in B$;

(b) subject to (a) P contains as many as possible vertices of B;

(c) subject to the above, P is as long as possible

Firstly we study several properties of the path P.

If there is a cycle C containing all the B-vertices on P, then it is clear that either C contains all B-vertices (and hence $|C| \geq |B|$) or there is another path containing more B-vertices than P. We assume that no such cycle exists.

If $v_i \in N(v_1) \cap P$, the cycle $P[v_1, v_{i-1}]v_i v_1$ is of length i. Thus there is a cycle of length at least $|N(v_1) \cap P| + 1$. Since $d_G(v_1) \geq k-1$, without loss of generality we assume that $S_1 := N(v_1) - P \neq \emptyset$ and similarly $S_p := N(v_p) - P \neq \emptyset$. By the choice of P and the independence of S, we have $S_1 \subseteq S$, $S_p \subseteq S$, $S_1 \cap S_p = \emptyset$, $N(S_1) \cup N(S_p) \subseteq B \cap P$, $(N(S_1) - \{v_1\})^- \cup (N(S_p) - \{v_p\})^+ \subset S$ and $S_1 \cap S_1^- = S_p \cap S_p^+ = \emptyset$.

We have

$$N(S_1)^- \cap N(S_p)^{+2} = \emptyset$$

since $N(S_1)^- \cup N(S_p)^+ \subseteq S$ and S is independent, and

$$(N(v_1) \cup N(S_1))^- \cap (N(S_p) \cup N(v_p)) = \emptyset$$

since otherwise there is a cycle containing $V(P)$, a contradiction. If $v_s \in N(S_1)^- \cap N(S_p)^+$, then there is a cycle containing $V(P) - \{v_s\}$, a contradiction because $v_s \in S$. It follows that

$$N(S_1)^- \cap N(S_p)^+ = \emptyset.$$

Since G is 2-connected, there exists a vine $Q := \{H_l[v_{i_l}, v_{j_l}] \ : \ 1 \leq l \leq m\}$ on the path $P(v_1, v_p)$, where $H_l[v_{i_l}, v_{j_l}]$ is a path between v_{i_l} and v_{j_l}, with all internal vertices in $G - P(v_1, v_p)$, such that $1 = i_1 < i_2 < j_1 \leq i_3 < j_2 \leq i_4 < ... \leq i_m < j_{m-1} < j_m = p$. We have the following cycles:

If m is even,

$$\begin{aligned} C_Q \ := \ & v_1 v_2 ... v_{i_2}^- H_2 P[v_{j_2}^+, v_{i_4}^-] H_4 P[v_{j_4}^+, v_{i_6}^-] ... v_m^- H_m \overline{P}[v_{p-1}, v_{j_{m-1}}^+] \\ & \overline{H}_{m-1} \overline{P}[v_{i_{m-1}}^-, v_{j_{m-3}}^+] ... \overline{P}[v_{i_3}^-, v_i^+] \overline{H}_1; \end{aligned}$$

and if m is odd,

$$\begin{aligned} C_Q &:= v_1v_2...v_{i_2}^- H_2 P[v_{j_2}^+, v_{i_4}^-] H_4 P[v_{j_4}^+, v_{i_6}^-]...v_{i_{m-1}}^- H_{m-1} P[v_{j_{m-1}}^+, v_{p-1}] \\ &\quad \overline{H}_m \overline{P}[v_{i_m}^-, v_{j_{m-2}}^+] \overline{H}_{m-2} \overline{P}[v_{i_{m-2}}^-, v_{j_{m-4}}^+]...\overline{P}[v_{i_3}^-, v_i^+] \overline{H}_1. \end{aligned}$$

Clearly we may choose the vine Q such that

$$(N(v_1) \cap P) \cup N(S_1) \subseteq P[v_1, v_{i_3}] \text{ and } (N(v_p) \cap P) \cup N(S_p) \subseteq P[v_{j_{m-2}}^+, v_p].$$

We will prove that $|C_Q| \geq k-3$.

Put

$$U_1 := \{v_1\} \cup N_P(v_1) \cup (N_P(v_p) - \{v_{i_m}\})^+.$$

and

$$U_2 := N(S_1) \cup (N(S_1) - \{v_1, v_{j_1}\})^- \cup (N(S_p) - \{v_p, v_{i_m}\})^+ \cup (N(S_p) - \{v_p, v_{i_m}, v_{j_{m-1}}\})^{+2}.$$

From the disjoint properties that we have obtained above, it follows that

$$|C_Q| \geq |U_1| = |N_P(v_1)| + |N_P(v_p)|$$

and

$$\begin{aligned} |C_Q| \geq |U_2| &\geq 2|N(S_1)| + 2|N(S_p)| - 6 \\ &= |S_1| + |S_p| - 4. \end{aligned}$$

These give

$$\begin{aligned} |C_Q| &\geq \tfrac{1}{2}(|U_1| + |U_2|) \\ &\geq \tfrac{1}{2}(d(v_1) + d(v_p) - 4) \\ &\geq k-3. \end{aligned}$$

□

<u>Lemma 2:</u> Let k be an integer with $3 \leq k \leq n-1$ and G a connected graph of order n such that

(a) there are at least $n/2 - 1$ vertices of degree at least k,

(b) all vertices of degree less than k are independent,

(c) any B-vertex is adjacent to at most one vertex of degree 1 and

(d) there does not exist a vertex v such that $G-v$ has at least two components containing vertex of degree at least k,

then either the circumference of G is at least k or $G = K^*_{\frac{n}{2}-1,\frac{n}{2}+1}$ (in this case $k = n-1$ and the circumference is $n-2$).

Proof of Lemma 2 : Again put $B = \{v : d(v) \geq k\}$ and $S = V(G) - B$.

Let H be the graph obtained from G by deleting all vertices of degree 1. We will show that there is a cycle of length at least $min\{|B|, k\}$ in H.

To prove this, without loss of generality we assume that H is a minimum counter-example. H is 2-connected and every B-vertex has degree at least $k-1$ in H. For any subset $S^* \subseteq S \cap H$, since every vertex in $B - N_H(S^*)$ is of degree at least $k-1$, by the minimality hypothesis, we get $|B - N_H(S^*)| < (|H - S^*|)/2 - 1$, which gives $|S^*| < 2|N_H(S^*)|$. By using Lemma 1, there is a cycle of length at least $min\{|B|, k\}$ in H.

Suppose that G does not contain a cycle of length at least k. Hence $k \geq |B|+1 \geq n/2$. By a theorem in [6] and [1] , G has a cycle containing all B-vertices. Let C be a longest cycle in G with $B \subseteq V(C)$. Let B_0 be the set of all B-vertices such that their processors in C belong to B. Since S is independent, $|B_0| = |B| - |S \cap C|$.

If two B_0-vertices b_1 and b_2 have a common neighbor $w \in S - C$, by the definition, b_1^+ and b_2^+ are in B and thus, have degrees at least $n/2$. By a traditional proof, we can get a longer cycle than C, a contradiction. So we assume that any pair of B_0-vertices have no common neighbor in $S - C$.

Let $b \in B_0$. Since b^+ has degree at least k and $|C| \leq k-1$, b^+ has some neighbor $s \in S - C$ with $d(s) \geq 2$ by (c). So from (b), s has a neighbor $b_1 \in C - \{b, b^+, b^{+2}\}$. By the maximality of C, we deduce b is not adjacent to b_1^-. So we assume that any B_0-vertex has at least one nonadjacency in C.

It follows from the above assumptions and $|C| \leq k-1$ that every B_0-vertex has at least three neighbors in $S-C$ and all these neighbors are different. So $|S-C| \geq 3(|B|-|S \cap C|)$. Also $|S| = |S-C|+|S \cap C| \geq 3|B|-2|S \cap C|$. Hence $|S \cap C| \geq |B|-1$ since $|B| \geq n/2-1$. We have $k-1 \geq |C| = |B| + |S \cap C| \geq 2|B| - 1 \geq n-3$.

If $k = n-1$ it is easy to deduce directly that $G = K^*_{\frac{n}{2}-1, \frac{n}{2}+1}$. If $k = n-2$, all the equalities holds in the above paragraph. From $|S \cap C| = |B|-1$, $B_0 = \{b\}$ for some $b \in B$. by the above augment, b has at least three neighbors in $S - C$. Similarly b^+ should have three neighbors in $S - C$. But $S - C$ has at most three vertices. It follows that b and b^+ have common neighbor in $S - C$ and C can be extended, a contradiction.

□

3 The main results

Proof of Theorem 2 : Let k and n be integers with $3 \leq k \leq n-1$. Suppose to the contrary, that there is a graph G of order n such that there are at least $n/2 - 1$ vertices of degree at least k and that the circumference of G is less than k and G has no subgraph isomophic to the graph $K^*_{\frac{k-1}{2}, \frac{k+3}{2}}$.

To get a contradiction, we just prove that G satisfies the conditions of Lemma 2. Without loss of generality, we assume that G is a minimum counter-example of the theorem. By the minimality, we may assume that G is connected and S is independent.

Suppose first that there exists some vertex v such that $G-v$ has at least two components H_1 and H_2 with $B_1 := H_1 \cap B \neq \emptyset$ and $B_2 := H_2 \cap B \neq \emptyset$. Put $G_1 := G[H_1 \cup \{v\}]$ and $G_2 := G[H_2 \cup \{v\}]$. Since $|B_1|+|B_2| \geq |B|-|\{v\}| \geq n/2-2 \geq \frac{1}{2}(|V(G_1)|+|V(G_2)|-1)-2 \geq \frac{1}{2}|V(G_1)|-1+\frac{1}{2}|V(G_2)|-1-\frac{1}{2}$ and hence at least one of G_1 and G_2, say G_1, has at least $\frac{1}{2}|V(G_1)|-1$ vertices of degree at least k. By the minimality hypothesis of G, either the circumference of G_1 is at least k or G_1 has a subgraph isomophic to the graph $K^*_{\frac{k-1}{2},\frac{k+3}{2}}$. Since G_1 is a subgraph of G, we have a contradiction. Therefore we assume that there does not exist a vertex v such that $G-v$ has at least two components containing vertex of degree at least k.

For any, $v \in B$ such that $S_0 := \{u \in S \cap N(v) : d(u) = 1\}$. Put $G_1 := G - S_0$. Clearly G_1 has at least $|B - \{v\}|$ vertices of degrees at least k. By the minimality of G, we deduce that $|B|-1 < \frac{1}{2}|G_1|-1 = \frac{1}{2}(n-|S_0|)-1$ and hence $|S_0| \leq 1$.

We have shown that G satisfies the conditions (a),(b),(c) and (d) of Lemma 2 and so by Lemma 2, either the circumference of G is at least k or G has a subgraph isomophic to the graph $K^*_{\frac{k-1}{2},\frac{k+3}{2}}$. This contradiction completes the proof.

□

Proof of Theorem 3 : Suppose that G is a 3-connected graph of order n such that at least $\frac{n}{2}+k$ vertices are of degree at least k, $k \geq 25$ and G does not contain a cycle of length at least $2k$. Denote by $B = \{u \in V(G) : d(u) \geq k\}$ and $S = V(G) - B$.

Let $C = c_1c_2...c_pc_1$ be a longest cycle in G. Since $G-C$ contains at least $\frac{n}{2}+k-(2k-1) = \frac{n}{2}-k+1 \geq |S|+1$ vertices in B. Hence there exists a component H of $G-C$ such that $|H \cap B| \geq |S \cap H|+1$. Let $d = k - max\{|N(u) \cap C| : u \in H \cap B\}$ and $N(u_f) \cap C = \{c_{m_1}, c_{m_2}, ..., c_{m_{k-d}}\} \subset C$. Then every vertex of $H \cap B$ has degree at least d in H.

By Theorem 2, either H admits a longest cycle C_H of $q \geq d$ vertices or H has a subgraph C_H isomophic to $K^*_{\frac{d-1}{2},\frac{d+3}{2}}$.

We claim that the longest cycle $C_H = u_1u_2...u_qu_1$ in H has at least 8 vertices.

If $H \cap B = \{u\}$ then $H = \{u\}$ and by the maximality of C, $|C| \geq 2d(u) \geq 2k$. Assume that $|H \cap B| \geq 2$ and $q \leq 7$. For any vertex $u \in H \cap B - \{u_f\}$, by the maximality of C, we have $|C| \geq |N_C(u_f)^+| + |N_C(u_f)|^{+2} + |N_C(u)| \geq 2(k-d) + |N_C(u)|$ and hence $|N_C(u)| \leq 2d-1$ and $|N_H(u)| \geq k-2d+1$. It follows that in the subgraph $H-\{u_f\}$, there are at least $\frac{|H-\{u_f\}|}{2}$ vertices of degree at least $k-2d$. By Theorem 2, $H-\{u_f\}$ has a cycle of at least $min\{k-2d-1, d-1\}$ vertices. Then $k-2d-1 \leq 7$ and $d-1 \leq 7$, contrary to $k \geq 25$. The claim holds.

Since G is 3-connected, there are three disjoint paths P_1, P_2, P_3 between three distinct vertices $c_i, c_j, c_m \in C$ and three distinct vertices $u_{i'}, u_{j'}, u_{m'} \in C_H$ respectively.

Assume first that $d \geq k-2$. By the maximality of C, if C_H is a cycle of $q \geq d$ vertices, we have $|C(c_i, c_j)| \geq |\overline{C_H}[u_{i'}, u_{m'}]\overline{C_H}(u_{m'}, u_{j'}]|$, $|C(c_j, c_m)| \geq |\overline{C_H}[u_{j'}, u_{i'}]\overline{C_H}(u_{i'}, u_{m'}]|$ and $|C(c_m, c_i)| \geq |\overline{C_H}[u_{m'}, u_{j'}]\overline{C_H}(u_{j'}, u_{i'}]|$.

$$\begin{aligned}
|C| &\geq |\{c_i, c_j, c_m\}| + |C(c_i, c_j)| + |C(c_j, c_m)| + |C(c_m, c_i)| \\
&\geq |\{c_i, c_j, c_m\}| + |\overline{C_H}[u_{i'}, u_{m'}]\overline{C_H}(u_{m'}, u_{j'}]| + |\overline{C_H}[u_{j'}, u_{i'}]\overline{C_H}(u_{i'}, u_{m'}]| + |\overline{C_H}[u_{m'}, u_{j'}]\overline{C_H}(u_{j'}, u_{i'}]| \\
&\geq 3 + 2|C_H(u_{i'}, u_{j'})| + 2|C_H(u_{j'}, u_{m'})| + 2|C_H(u_{m'}, u_i')| + 3|\{u_{i'}, u_{j'}, u_{m'}\}| \\
&\geq 3 + 2|C_H| + 3 \\
&\geq 2k.
\end{aligned}$$

When $C_H = K^*_{\frac{d-1}{2}, \frac{d+3}{2}}$, then clearly $|C(c_i, c_j)| \geq d-2$, $|C(c_j, c_m)| \geq d-2$ and $|C(c_m, c_i)| \geq d-2$. It follows that when $k \geq 9$,

$$\begin{aligned}
|C| &\geq |\{c_i, c_j, c_m\}| + |C(c_i, c_j)| + |C(c_j, c_m)| + |C(c_m, c_i)| \\
&\geq |\{c_i, c_j, c_m\}| + 3(d-2) \\
&\geq 3k-9 \\
&\geq 2k.
\end{aligned}$$

Then we assume that $d \leq k-3$. Then clearly $|C(c_{m_g}, c_{m_{g+1}})| \geq 1$ for any g.

Without loss of generality we may choose the paths P_1, P_2, P_3 such that if $u_f \in C_H$, $u_f = u_{m'}$ and if $u_f \notin C_H$, there is a path P_4 between u_f and the vertex $u_{m'}$ such that $P_3 = P_4[u_{m'}, u_f)u_f c_m$.

When $C(c_i, c_j) \cap N(u_f) \neq \emptyset$ and $\overline{C}(c_j, c_i) \cap N(u_f) \neq \emptyset$, let $c_{m_{h'}}, c_{m_g} \in N(u_f) \cap C(c_i, c_j)$ and $c_{m_{g'}}, c_{m_h} \in N(u_f) \cap C(c_j, c_i)$ such that $(C(c_{m_h}, c_i) \cup C(c_i, c_{m_{h'}})) \cap (N(u_f) \cup \{c_j\}) = \emptyset$ and $(C(c_{m_g}, c_j) \cup C(c_j, c_{m_{g'}})) \cap (N(u_f) \cup \{c_i\}) = \emptyset$ (*i.e.*, c_{m_h} is the last vertex of $N(u_f) \cap C$ before c_i, $c_{m_{h'}}$ is the first vertex of $N(u_f) \cap C$ after c_i, c_{m_g} is the last vertex of $N(u_f) \cap C$ before c_j) and $c_{m_{g'}}$ is the first vertex of $N(u_f) \cap C$ after c_j). If C_H is a cycle, by the maximality of C, we have $|C(c_{m_g}, c_j))| \geq |C_H[u_{m'}, u_{i'}]C_H(u_{i'}, u_{j'}]|$ and $|C(c_j, c_{m_{g'}})| \geq |\overline{C_H}[u_{m'}, u_{j'}]|$. These give $|C(c_{m_g}, c_{m_{g'}})| \geq |C_H| + 3$. Similarly we have $|C(c_{m_h}, c_{m_{h'}})| \geq |C_H| + 3$. It follows that when $q \geq d$

$$\begin{aligned}
|C| &\geq |N(u_f)| - 2 + |N(u_f)| - 4 + 2(|C_H| + 3) \\
&\geq 2(k-d) - 6 + 2q + 6 \\
&\geq 2k,
\end{aligned}$$

a contradiction. It follows that $8 \leq q \leq d-1$ and $C_H = K^*_{\frac{d-1}{2}, \frac{d+3}{2}}$. Clearly $|C(c_{m_h}, c_i)| \geq d-2$, $|C(c_i, c_{m_{h'}})| \geq d-2$ $|C(c_{m_g}, c_j))| \geq d-2$ and $|C(c_j, c_{m_{g'}})| \geq d-2$. Then we obtain

$$\begin{aligned}
|C| &\geq |N(u_f)| + |N(u_f)| - 4 + 4(d-2) \\
&\geq 2(k-d) - 4 + 4d - 8 \\
&\geq 2k + 2d - 12 \\
&> 2k,
\end{aligned}$$

a contradiction.

Assume then that at least one of $C(c_i, c_j) \cap N(u_f)$ and $\overline{C}(c_j, c_i) \cap N(u_f)$, say $C(c_i, c_j) \cap N(u_f) = \emptyset$.

let $c_{m_h}, c_{m_g} \in N(u_f) \cap C(c_j, c_i)$ such that $(C(c_{m_h}, c_i) \cup C(c_j, c_{m_g})) \cap N(u_f) = \emptyset$ (*i.e.*, c_{m_h} is the last vertex of $N(u_f) \cap C$ before c_i, c_{m_g} is the first vertex of $N(u_f) \cap C$ after c_j).

Let $C(c_{m_h}, c_i) \neq \emptyset$ and $C(c_{m_h}, c_i) \cap (N(u_f) \cup \{c_j\}) = \emptyset$ (*i.e.*, c_{m_h} is the last vertex of $N(u_f) \cap C$ before c_i) and let $C(c_j, c_{m_l}) \neq \emptyset$ and $C(c_j, c_{m_l}) \cap (N(u_f) \cup \{c_i\}) = \emptyset$ (*i.e.*, c_{m_l} is the first vertex of $N(u_f) \cap C$ after c_j).

If C_H is a cycle, by the maximality of C, we have $|C(c_{m_h}, c_i))| \geq |C_H(u_{i'}, u_{j'})C_H[u_{j'}, u_{m'}]|$, $|C(c_i, c_j)| \geq |\overline{C_H}[u_{i'}, u_{m'})\overline{C_H}[u_{m'}, u_{j'}]|$ and $|C(c_j, c_{m_g})| \geq |\overline{C_H}[u_{j'}, u_{i'})\overline{C_H}[u_{i'}, u_{m'}]|$. These give

$$\begin{aligned} |C| &\geq |N(u_f)| + |N(u_f)| - 3 + 2|C_H| + 3 \\ &\geq 2(k-d) + 2q, \end{aligned}$$

a contradiction when $q \geq d$. It implies that $8 \leq q \leq d-1$ and $C_H = K^*_{\frac{d-1}{2}, \frac{d+3}{2}}$. Since $d \geq 9$ and $|C(c_{m_h}, c_i))| \geq d-2$, $|C(c_i, c_j)| \geq d-2$ and $|C(c_j, c_{m_g})| \geq d-2$, we obtain

$$\begin{aligned} |C| &\geq |N(u_f)| + |N(u_f)| - 3 + 3(d-2) \\ &\geq 2(k-d) + 3d - 9 \\ &\geq 2k + d - 9 \\ &\geq 2k, \end{aligned}$$

a contradiction.

The proof is complete.

□

4 Appendix: Proof of Lemma 1

For any vertex v and a condition A, let $\theta(v : A) = \{v\}$ if A is satisfied or $\theta(v : A) = \emptyset$ if A is not satisfied.

Proof of Lemma 1 : Let $P := v_1v_2...v_p$ be a path in G such that

(a) $v_1, v_p \in B$;

(b) subject to (a) P contains as many as possible vertices of B;

(c) subject to the above, P is as long as possible and

(d) subject to the above, $max\{i : v_iv_1 \in E(G)\}$ is as large as possible.

Firstly we study several properties of the path P.

If there is a cycle C containing all the B-vertices on P, then it is clear that either C contains all B-vertices (and hence $|C| \geq |B|$) or there is another path containing more B-vertices than P. We assume that no such cycle exists.

If $v_i \in N(v_1) \cap P$, the cycle $P[v_1, v_{i-1}]v_iv_1$ is of length i. Thus there is a cycle of length at least $|N(v_1) \cap P| + 1$. Since $d_G(v_1) \geq k-1$, without loss of generality we assume that $S_1^0 := N(v_1) - P \neq \emptyset$ and similarly $S_p^0 := N(v_p) - P \neq \emptyset$. Put $S_1 := S_1^0 \cup \theta(v_2 : v_2 \in N(S_1^0)^-)$ and $S_p := S_p^0 \cup \theta(v_{p-1} : v_{p-1} \in N(S_p^0)^+)$. By the choice of P and the independence of S, we have $S_1 \subseteq S$, $S_p \subseteq S$, $S_1 \cap S_p = \emptyset$, $N(S_1) \cup N(S_p) \subseteq B \cap P$, $(N(S_1) - \{v_1\})^- \cup (N(S_p) - \{v_p\})^+ \subset S$ and $S_1 \cap S_1^- = S_p \cap S_p^+ = \emptyset$.

We have

$$N(S_1)^- \cap N(S_p)^{+2} = \emptyset$$

since $N(S_1)^- \cup N(S_p)^+ \subseteq S$ and S is independent, and

$$(N(v_1) \cup N(S_1))^- \cap (N(S_p) \cup N(v_p)) = \emptyset$$

since otherwise there is a cycle containing $V(P)$, a contradiction. If $v_s \in N(S_1)^- \cap N(S_p)^+$, then there is a cycle containing $V(P) - \{v_s\}$, a contradiction because $v_s \in S$. It follows that

$$N(S_1)^- \cap N(S_p)^+ = \emptyset.$$

For any $w^* \in S_1 \cap N(v_3)$ and $w^{**} \in S_p \cap N(v_{p-2})$, define a path $P_{(w^*,w^{**})} := v_1w^*v_3v_4...v_{p-2}w^{**}v_p$ which has the same properties as P.

Since G is 2-connected, there exists a vine $Q := \{H_l[v_{i_l}, v_{j_l}] \ : \ 1 \leq l \leq m\}$ on the path $P_{(w^*,w^{**})}$, where $H_l[v_{i_l}, v_{j_l}]$ is a path between v_{i_l} and v_{j_l}, with all internal vertices in $G - P_{(w^*,w^{**})}$, such that $1 = i_1 < i_2 < j_1 \leq i_3 < j_2 \leq i_4 < ... \leq i_m < j_{m-1} < j_m = p$. We have the following cycles:
If m is even,

$$\begin{aligned} C_Q \ := \ & v_1w^*...v_{i_2}^-H_2P[v_{j_2}^+, v_{i_4}^-]H_4P[v_{j_4}^+, v_{i_6}^-]...v_m^-H_m\overline{P}[w^{**}, v_{j_{m-1}}^+] \\ & \overline{H}_{m-1}\overline{P}[v_{i_{m-1}}^-, v_{j_{m-3}}^+]...\overline{P}[v_{i_3}^-, v_i^+]\overline{H}_1; \end{aligned}$$

and if m is odd,

$$\begin{aligned} C_Q \ := \ & v_1w^*...v_{i_2}^-H_2P[v_{j_2}^+, v_{i_4}^-]H_4P[v_{j_4}^+, v_{i_6}^-]...v_{i_{m-1}}^-H_{m-1}P[v_{j_{m-1}}^+, w^{**}] \\ & \overline{H}_m\overline{P}[v_{i_m}^-, v_{j_{m-2}}^+]\overline{H}_{m-2}\overline{P}[v_{i_{m-2}}^-, v_{j_{m-4}}^+]...\overline{P}[v_{i_3}^-, v_i^+]\overline{H}_1. \end{aligned}$$

We note that in the above cases, the paths H_1 and H_m are contained in the cycles.

We may choose a vine Q $(N(v_1) \cap P) \cup N(S_1) \subseteq P[v_1, v_{i_3}]$ and $(N(v_p) \cap P) \cup N(S_p) \subseteq P[v_{j_{m-2}}^+, v_p]$.

Let v^* be the first vertex on P that is adjacent to v_p. Put

$$\begin{aligned} U_1 \ := \ & \{v_1\} \cup N_P(v_1) \cup (N_P(v_p) - \{v_{i_m}\})^+ \cup \theta(v_{j_{m-1}} : v_pv_{j_{m-1}}^- \notin E(G) \text{ and } v_1v_{j_{m-1}} \notin E(G)) \\ & \cup\theta(v^* : v^* \notin N(v_1)) \cup \theta(w_1 : v_{j_1} \in N(S_1)) \cup \theta(w_2 : v_{i_m} \in N(S_p)) \end{aligned}$$

and

$$\begin{aligned}U_2 \;:=\;& N(S_1)\cup((N(S_1)-\{v_1\})^- -\theta(v_{j_1-1}:v_{j_1}\in N(S_1))\\ &\cup((N(S_p)-\{v_p\})^+ -\theta(v_{i_m+1}:v_{i_m}\in N(S_p)-\{v^-_{j_{m-1}}\}))\cup((N(S_p)-\{v_p\})^{+2}\\ &-\theta(v_{i_m+1}:v_{i_m-1}\in N(S_p)-\{v^{-2}_{j_{m-1}}\})-\theta(v_{i_m+2}:v_{i_m}\in N(S_p)-\{v^-_{j_{m-1}},v^{-2}_{j_{m-1}}\}))\\ &\cup\theta(w_1:v_{j_1}\in N(S_1))\cup\theta(w_2:v_{i_m}\in N(S_p))\cup\theta(v_2:v_2\notin N(S^0_1)^-)\\ &\cup\theta(v_p:v_{p-2}\notin N(S^0_p)),\end{aligned}$$

where $w_1\in N(v_{j_1})\cap N(S^0_1)$ and $w_2\in N(v_{i_m})\cap N(S^0_p)$.

It follows that

$$\begin{aligned}|C_Q| \;\geq\;& |U_1| = |N_P(v_1)|+|N_P(v_p)|+|\theta(v_{j_{m-1}}:v_pv^-_{j_{m-1}}\notin E(G)\text{ and }v_1v_{j_{m-1}}\notin E(G))|\\ &+|\theta(v^*:v^*\notin N(v_p))|+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|.\end{aligned}$$

Since $N(S_p)\cap N(S_p)^+=\emptyset$, $|\theta(v_{i_m+1}:v_{i_m-1}\in N(S_p)-\{v^{-2}_{j_{m-1}}\})|+|\theta(v_{i_m+2}:v_{i_m}\in N(S_p)-\{v^-_{j_{m-1}},v^{-2}_{j_{m-1}}\})|\leq 1$. Because $|N(v_1)-P|=|S_1|-|\theta(v_2:v_2\in N(N(v_1)-P)^-)|$ and $|N(v_p)-P|=|S_p|-|\theta(v_{p-1}:v_{p-1}\in N(N(v_p)-P)^+)|$, we obtain

$$\begin{aligned}|C_Q| \;\geq\;& |U_2|\\ =\;& 2|N(S_1)|-1-|\theta(v_{j_1-1}:v_{j_1}\in N(S_1))|+2|N(S_p)|-2-|\theta(v_{i_m+1}:v_{i_m}\in N(S_p))|\\ &-|\theta(v_{i_m+1}:v_{i_m-1}\in N(S_p)-\{v^{-2}_{j_{m-1}}\})|-|\theta(v_{i_m+2}:v_{i_m}\in N(S_p)-\{v^-_{j_{m-1}},v^{-2}_{j_{m-1}}\}))|\\ &+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|+|\theta(v_2:v_2\notin N(S^0_1)^-)|\\ &+|\theta(v_p:v_{p-1}\notin N(S^0_p)^+)|\\ \geq\;& |S_1|+|S_p|-2-|\theta(v_{j_1-1}:v_{j_1}\in N(S_1))|-|\theta(v_{i_m+1}:v_{i_m}\in N(S_p))|\\ &+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|+|\theta(v_2:v_2\notin N(S^0_1)^-)|\\ &+|\theta(v_p:v_{p-1}\notin N(S^0_p)^+)|\\ =\;& |N(v_1)-P|+|N(v_p)-P|-2+|\theta(v_2:v_2\in N(N(v_1)-P)^-)|\\ &+|\theta(v_{p-1}:v_{p-1}\in N(N(v_p)-P)^+)|+|\theta(v_2:v_2\notin N(S^0_1)^-)|+|\theta(v_p:v_{p-1}\notin N(S^0_p)^+)|\\ =\;& |N(v_1)-P|+|N(v_p)-P|.\end{aligned}$$

It follows that

$$\begin{aligned}|C_Q| \;\geq\;& \tfrac{1}{2}(|U_1|+|U_2|)\\ \geq\;& \tfrac{1}{2}(|N(v_1)-P|+|N(v_p)-P|+|N_P(v_1)|+|N_P(v_p)|+|\theta(v^*:v^*\notin N(v_1))|\\ &+|\theta(v_{j_{m-1}}:v_pv^-_{j_{m-1}}\notin E(G)\text{ and }v_1v_{j_{m-1}}\notin E(G))|+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|)\\ \geq\;& \tfrac{1}{2}(d(v_1)+d(v_p)+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|\\ &+|\theta(v^*:v^*\notin N(v_1))|+|\theta(v_{j_{m-1}}:v_pv^-_{j_{m-1}}\notin E(G)\text{ and }v_1v_{j_{m-1}}\notin E(G))|)\\ \geq\;& k-1+\tfrac{1}{2}(|\theta(v^*:v^*\notin N(v_1))|+|\theta(v_{j_{m-1}}:v_pv^-_{j_{m-1}}\notin E(G)\text{ and }v_1v_{j_{m-1}}\notin E(G))|\\ &+|\theta(w_1:v_{j_1}\in N(S_1))|+|\theta(w_2:v_{i_m}\in N(S_p))|).\end{aligned}$$

Then we assume that $|C_Q|=k-1$ and deduce that $C_Q=U_1=U_2$, $v^*\in N(v_1)\cap N(v_p)$ (which implies $2\leq m\leq 3$), $v_{j_1}\notin N(S_1)$, $v_{i_m}\notin N(S_p)$ and either $v_pv^-_{j_{m-1}}\in E(G)$ or $v_1v_{j_{m-1}}\in E(G)$. One of $v^-_{i_m}$ and $v^{-2}_{i_m}$ is in $N(S_p)$. By symmetric, we may get either $v_1v^+_{i_2}\in E(G)$ or $v_pv_{i_2}\in E(G)$. If $m=3$, then $v_1v_{j_2}\notin E(G)$ and $v_pv_{i_2}\notin E(G)$. Thus $v_pv^-_{j_2}\in E(G)$ and $v_1v^+_{i_2}\in E(G)$. It implies that $V(P)\subseteq C_Q$, a contradiction. So we assume that $m = 2$. It follows that $v_{j_1}^- v_p \notin E(G)$ and hence $v_1 v_{j_1} \in E(G)$. Similarly $v_p v_{i_2} \in E(G)$.

If $v_{i_2}^{-2} \in N(S_p)$, then $v_{i_2} \notin N(v_1)$ and since $V(C_Q) = U_1$, $v_{i_2}^- \in N(v_p)$. Then let $w^* \in S_p \cap N(v_{i_2}^{-2})$ and put

$$P^* = P[v_1, v_{i_2}^{-2}] w^* v_p v_{i_2}^- P[v_{i_2}, v_{p-2}]\{v_{p-1}\}.$$

If $v_{i_2}^- \in N(S_p)$. Let $w^* \in S_p \cap N(v_{i_2}^-)$ and put

$$P^{**} = P[v_1, v_{i_2}^{-2}] v_{i_2}^- w^* v_p v_{i_2} P(v_{i_2}, v_{p-2}]\{v_{p-1}\}.$$

Where v_{p-1} is in P^* or P^{**} if and only if it is in $B - \{w^*\}$. P^* and P^{**} satisfy the hypotheses (a)(b) and (c), but are contrary to (d) because $N(v_1) \cap P[v_{j_1}, v_{p-2}] \neq \emptyset$. □

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