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Partition of a Graph into Cycles and Degenerated Cycles

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Abstract

Let G be a graph of order n and k any positive integer with $k \leq n$. It has been shown by Brandt *et al.* that if $|G| = n \geq 4k$ and if the degree sum of any pair of nonadjacent vertices is at least n, then G can be partitioned into k cycles. We prove that if the degree sum of any pair of nonadjacent vertices is at least $n - k + 1$, then G can be partitioned into k subgraphs H_i, $1 \leq i \leq k$, where H_i is a cycle or K_1 or K_2, except $G = C_5$ and $k = 2$.

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1 Introduction

In this paper, we only consider finite undirected graphs without loops and multiple edges. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. With a slight abuse of notation, for a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S \rangle$, and $G - S = \langle V(G) - S \rangle$. For a graph G, $|V(G)|$ is the order of G, $\delta(G)$ is the minimum degree of G, and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) | x, y \in V(G), x \neq y, xy \notin E(G)\}$$

is the minimum degree sum of nonadjacent vertices. (When G is a complete graph, we define $\sigma_2(G) = \infty$.)

In this paper, "disjoint" means "vertex-disjoint," since we only deal with partitions of the vertex set.

Suppose $H_1, \cdots, H_k$ are disjoint subgraphs of G such that $V(G) = \bigcup_{i=1}^{k} V(H_i)$ and H_i is a cycle for all i, $1 \leq i \leq k$. Then the union of these H_i is a 2-factor of G with k components. A sufficient condition for the existence of a 2-factor with a specified number of components was given by Brandt et al. [1].

Theorem 1 *Suppose $|G| = n \geq 4k$ and $\sigma_2(G) \geq n$. Then G can be partitioned into k cycles, that is, G contains k disjoint cycles $H_1, \cdots, H_k$ satisfying $V(G) = \bigcup_{i=1}^{k} V(H_i)$.*

In fact the first author pointed out in [4] that Theorem 1 holds for $n \geq 4k - 1$ (and this is sharp).

In this paper, we show that weaker conditions than Theorem 1 are suffincient if we regard K_1 and K_2 as degenerated cycles.

Theorem 2 *Let G be a graph of order n and k any positive integer with $k \leq n$. If $\sigma_2(G) \geq n - k + 1$, then G can be partitioned into k subgraphs H_i, $1 \leq i \leq k$, where H_i is a cycle or K_1 or K_2, except $G = C_5$ and $k = 2$.*

Proof: We first have

Claim 1: If G has $n-k$ independent edges or $k=1$, then there is a desired partition.

Proof: If G has $n-k$ independent edges, by using these edges and the remained individual vertices of G we have a desired partition. If $k=1$, by Ore 's theorem, either G has a hamiltonian cycle or $G=K_2$ or $G=K_1$. Hence we also have a desired partition. $\square$

By Claim 1, we assume $k \geq 2$. On the other hand, we can assume $n>k$ since, for $n=k$, the trivial partition of $V(G)$ is convenient. When $n=k+1$, the result is obvious as soon as G has at least one edge, which is the case since $\sigma_2(G) \geq 2$. Suppose $n=k+2$. Then $\sigma_2(G) \geq 3$ and hence there is some vertex w having two distinct neighbors u and v. If $uv \in E(G)$, G is partitioned into a triangle and $n-3=k-1$ K_1's. If $uv \notin E(G)$, one of u and v, say u, has a neighbor $w_1 \in G-\{w,v,u\}$. Then wv and uw_1 are independent edges. Hence we may suppose that $n \geq k+3$. Note that $n \geq 5$, since $k \geq 2$. In the rest of the proof, we use induction on n. That is, we assume that the conclusion is true for all graphs with at most $n-1$ vertices. We assume that the graph G of order n satisfies the hypothesis of the theorem and has no required partition.

Since $\sigma_2(G) \geq n-k+1 \geq 4$, G contains a cycle.

Claim 2: There exists a cycle with length at most $n-k+1$.

Proof: Suppose to the contrary that there is no such cycle in G. Let C be a shortest cycle. Then $|C| \geq n-k+2 \geq 5$ and C has no chord. If $n-k+1=4$, then $n=k+3$. Since G has no three independent edges, $|C|=5$. If $n=5$, this is the exceptional case. Suppose $n \geq 6$. Then there exists some vertex $w \in V(G-C)$. Take any vertex $v \in V(C)$. If $d_G(v) \geq 3$, G contains three independent edges. If $d_G(v)=2$, v and w are nonadjacent, and $d_G(w)+d_G(v) \geq \sigma_2(G) \geq 4$. So, $d_G(w)>0$, and again G contains three independent edges.

It follows that $n-k+1 \geq 5$ and there are at least $|C|-2$ vertices in C that has degree at least 3 and hence has some adjacency in $G-C$. By the minimality of C, any two vertices do not have a common neighbor in $G-C$. So there are distinct vertices $u_1, u_2, \dots, u_{|C|-2}$ in $G-C$ adjacent to distinct

$v_1, v_2, \ldots, v_{|C|-2}$ in C respectively (*i.e.* $u_i v_i \in E(G)$, $1 \leq i \leq |C|-2$). This means that there are $|C|-2$ independent edges. Since $|C|-2 \geq n-k$, the claim follows by Claim 1. $\square$

Choose a cycle $C = c_1 c_2 \ldots c_p c_1$ verifying the following conditions:

(1) $p \leq n-k+1$, and

(2) subject to (1), C is as long as possible.

We put the induced subgraph $R_1 = G - C$. Then $|R_1| = n - p \geq k-1$. If $|R_1| = k-1$, the cycle C and the $k-1$ vertices of R_1 give a partition of G. The required partition when $|R_1| = k$ and R_1 contains at least one edge is given by an edge of R_1, the $k-2$ remaining vertices of R_1 and C. When $|R_1| = k$ and R_1 is independent, by the maximality of C any vertex in R_1 has no consecutive neighbors in C and hence any pair of vertices in R_1 have degree sum at most p. But $p \geq \sigma_2(G) \geq n-k+1$ implies that $|R_1| \leq k-1$. So without loss of generality we assume $|R_1| \geq k+1$. Then $p \leq n-k-1$.

By the maximality of C, it is clear that every vertex w in R_1 does not have two consecutive neighbors in C and hence $|N_C(w)| \leq \frac{p}{2}$. If $\sigma_2(R_1) \geq |R_1| - (k-1) + 1$, then by the induction hypothesis, either R_1 can be partitioned into $k-1$ subgraphs isomorphic to a cycle or K_1 or K_2, or $R_1 = C_5$ and $k-1=2$. In the former case, we have a required partition together with C. In the latter case, let $R_1 = w_1 w_2 \ldots w_5 w_1$. If there is some vertex $c \in C \cap N(w_1)$ then to avoid a required k partition, w_3 is not adjacent to the successor and the second successor of c in C. Since w_1 has no consecutive neighbors in C, we deduce $|C| \geq 2d_C(w_1) + d_C(w_3)$ and similarly $|C| \geq 2d_C(w_3) + d_C(w_1)$. So $n-2 \leq d(w_1) + d(w_3) \leq 4 + \frac{2}{3}|C| = 4 + \frac{2}{3}(n-5)$ and thus $n \leq 8$. This gives that C is a triangle. We get a contradiction to (2). So we may assume that $\sigma_2(R_1) < |R_1| - (k-1) + 1$. Therefore there is a pair of nonadjacent vertices u_1 and u_2 in R_1 such that

$$d_{R_1}(u_1) + d_{R_1}(u_2) \leq |R_1| - k + 1 = n - p - k + 1$$

and hence

$$d_C(u_1) + d_C(u_2) \geq \sigma_2(G) - (n-p-k+1) \geq p.$$

Since a vertex of R_1 has at most $p/2$ neighbors in C,

$$d_C(u_1) = d_C(u_2) = \tfrac{p}{2} \;\; \textit{and} \;\; d_{R_1}(u_1) + d_{R_1}(u_2) = n-p-k+1.$$

Assume that $N(u_1)\cap C=\{c_1,c_3,c_5,...,c_{p-1}\}$. Let $H=\{w\in R_1 \mid N(w)\cap C=\{c_1,c_3,c_5,...,c_{p-1}\}\}$ and $R_2=R_1-H$.

Claim 3: If a vertex x in R_2 has at least one neighbor in $H\cup\{c_2,c_4,c_6,...,c_p\}$, then $|N(x)\cap(H\cup C)|=1$.

Proof: If the claim is false (*i.e.* $|N(x)\cap(H\cup C)|\geq 2$), it is easy to get a cycle of length $|C|+1$ or $|C|+2$ which is impossible because of the maximality hypothesis of C and $|R_1|\geq k+1$. □

This implies that $u_2\in H$.

Claim 4: $H\cup\{c_2,c_4,c_6,...,c_p\}$ is independent.

Proof: If there is an edge between two vertices in $H\cup\{c_2,c_4,c_6,...,c_p\}$, then we can easily get a cycle of length $|C|+1$ which is a contradiction. □

Claim 5: $|H|\leq k-1$. And if $|H|=k-1$, then $d_{R_1}(u_1)+d_{R_1}(u_2)=|R_2|$.

Proof: Since $u_1,u_2\in H$, by Claims 3 and 4,

$$n-k+1=d(u_1)+d(u_2)=p+d_{R_1}(u_1)+d_{R_1}(u_2)\leq p+|R_2|=n-|H|.$$

We deduce that $|H|\leq k-1$ and and if $|H|=k-1$, then $d_{R_1}(u_1)+d_{R_1}(u_2)=|R_2|$. □

Note that $|R_2|=|R_1|-|H|\geq k+1-|H|\geq 2$. If there is a pair of nonadjacent vertices u_3 and u_4 in R_2 such that

$$d_{R_2}(u_3)+d_{R_2}(u_4)\leq|R_2|-(k-1-|H|)=n-p-k+1,$$

then

$$d_{C\cup H}(u_3)+d_{C\cup H}(u_4)\geq\sigma_2(G)-(n-p-k+1)\geq p.$$

By Claim 3, it gives that $d_{C\cup H}(u_3)=d_{C\cup H}(u_4)=\frac{p}{2}$. By Claim 3 again and the maximality of C, we deduce that $N(u_3)\cap C=N(u_4)\cap C=\{c_1,c_3,c_5,...,c_{p-1}\}$, contrary to the definition of H. It follows that $\sigma_2(R_2)\geq|R_2|-(k-1-|H|)+1$.

Suppose $|H| \leq k-2$. By the induction hypothesis, either R_2 can be partitioned into $k-1-|H|$ subgraphs isomorphic to a cycle or a degenerated cycle, or $R_2 = C_5$ and $k-1-|H| = 2$. In the first case, together with C and the individual vertices in H, we have a required partition of G. Therefore $R_2 = C_5$ and $k-1-|H| = 2$. Put $C_2 = c_1u_1c_5c_6...c_pc_1$. Then R_2, C_2, c_2c_3, c_4 and the vertices in $H - \{u_1\}$ give a partition of $4+|H|-1 = k$ required subgraphs of G. So we have $|H| = k-1$.

Since $\sigma_2(R_2) \geq |R_2|+1$ and $|R_2| \geq 2$, R_2 is hamiltonian or $R_2 = K_2$. Let $C_3 = x_1x_2...x_qx_1$ be a hamiltonian cycle of R_2 or $C_3 = x_1x_2$. When u_1 has two consecutive neighbors in C_3, by adding u_1 between these two neighbors, we get a cycle C_3' and we have a partition with C, C_3' and the $k-2$ vertices in $H - \{u_1\}$. So u_1 (similarly for u_2) has no consecutive neighbors in C_3. By Claims 3 and 5, there are consecutive vertices x_i, x_{i+1} such that, without loss of generality, $x_iu_1, x_{i+1}u_2 \in E(G)$. Put $C_4 = c_1u_2x_{i+1}x_{i+2}...x_1x_2...x_iu_1c_5c_6...c_pc_1$. Then C_4, the edge c_3c_4 and the $|H|-1$ independent vertices of $(H \cup \{c_2\}) - \{u_1, u_2\}$ give a required partition of G.

The proof of the theorem is complete. □ □

References

[1] S.Brandt, G.Chen, R.Faudree, R.J.Gould and L.Lesniak, Degree conditions for 2-factors, J. Graph Theory 24 (1997) 165-173.

[2] G. Chartrand, L. Lesniak, *Graphs & Digraphs* (third edition), Chapman & Hall, London, 1996.

[3] Y.Egawa, R.J.Faudree, E.Győri, Y.Ishigami, R.H.Schelp and H.Wang, Vertex-disjoint cycles containing specified edges, Graphs and Combinatorics, 16 (2000). 81-92.

[4] H.Enomoto, On the existence of disjoint cycles in a graph, Combinatorica 18 (1998) pp. 487-492.

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